

GOVERNMENT POLYTECHNIC
NAYAGARH

LECTURE NOTES ON
CIRCUIT NETWORK THEORY

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CHAPTER 1

Magnetic Circuits

Introduction :

Magnetic flux lines always form closed loops. The closed path followed by the flux lines is called a magnetic circuit. Thus, a magnetic circuit provides a path for magnetic flux, just as an electric circuit provides a path for the flow of electric current. In general, the term magnetic circuit applies to any closed path in space, but in the analysis of electro-mechanical and electronic system this term is specifically used for circuits containing a major portion of ferromagnetic materials. The study of magnetic circuit concepts is essential in the design, analysis and application of electromagnetic devices like transformers, rotating machines, electromagnetic relays etc.

Magnetomotive Force (M.M.F) :

Flux is produced round any current – carrying coil. In order to produce the required flux density, the coil should have the correct number of turns. The product of the current and the number of turns is defined as the coil magnetomotive force (m.m.f),

If I = Current through the coil (A)

N = Number of turns in the coil.

Magnetomotive force = Current \times turns

So $M.M.F = I \times N$

The unit of M.M.F. is ampere–turn (AT) but it is taken as Ampere (A) since N has no dimensions.

Magnetic Field Intensity:

Magnetic Field Intensity is defined as the magneto-motive force per unit length of the magnetic flux path. Its symbol is H .

$$\text{Magnetic field Intensity (H)} = \frac{\text{Magnetomotive force}}{\text{Mean length of the magnetic path}}$$

$$\Rightarrow H = \frac{F}{l} = \frac{I.N}{l} \text{ A/m}$$

Where l is the mean length of the magnetic circuit in meters. Magnetic field intensity is also called magnetic field strength or magnetizing force.

Permeability :-

Every substance possesses a certain power of conducting magnetic lines of force. For example, iron is better conductor for magnetic lines of force than air (vacuum). Permeability of a material (μ) is its conducting power for magnetic lines of force. It is the ratio of the flux density. (B) Produced in a material to the magnetic field strength (H).

$$\text{i.e. } \mu = \frac{B}{H}$$

Reluctance :

Reluctance (S) is akin to resistance (which limits the electric Current).

Flux in a magnetic circuit is limited by reluctance. Thus reluctance(S) is a measure of the opposition offered by a magnetic circuit to the setting up of the flux.

Reluctance is the ratio of magnetomotive force to the flux. Thus

$$S = \frac{\text{Mmf}}{\phi}$$

Its unit is ampere turns per webber (or AT/wb).

Permeance:-

The reciprocal of reluctance is called the permeance (symbol A).

Permeance (A) = $1/S$ wb/AT

Turn T has no unit.

Hence permeance is expressed in wb/A or Henrys(H).

B.H. Curve :

Place a piece of an unmagnetised iron bar AB within the field of a solenoid to magnetise it. The field H produced by the solenoid, is called magnetising field, whose value can be altered (increased or decreased) by

changing (increasing or decreasing) the current through the solenoid. If we increase slowly the value of magnetic field (H) from zero to maximum value, the value of flux density (B) varies along 1 to 2 as shown in the figure and the magnetic materials (i.e iron bar) finally attains the maximum value of flux density (B_m) at point 2 and thus becomes magnetically saturated.

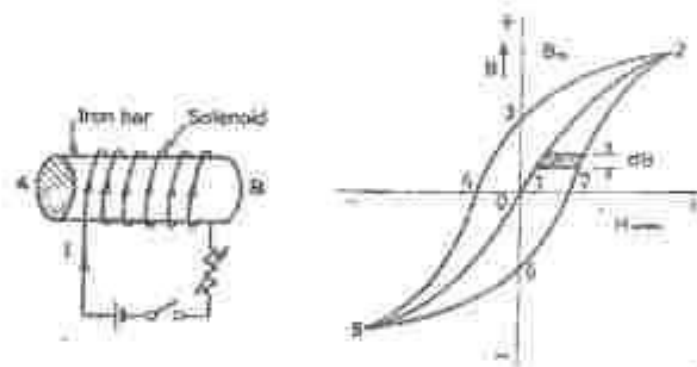


Fig. 2.1

Now if value of H is decreased slowly (by decreasing the current in the solenoid) the corresponding value of flux density (B) does not decrease along 2-1 but decreases somewhat less rapidly along 2 to 3. Consequently during the reversal of magnetization, the value of B is not zero, but is ' 13 ' at $H=0$. In other words, during the period of removal of magnetization force (H), the iron bar is not completely demagnetized.

In order to demagnetise the iron bar completely, we have to supply the demagnetisation force (H) in the opposite direction (i.e. by reversing the direction of current in the solenoid). The value of B is reduced to zero at point 4, when $H=14$. This value of H required to clear off the residual magnetisation, is known as coercive force i.e. the tenacity with which the material holds to its magnetism.

If after obtaining zero value of magnetism, the value of H is made more negative, the iron bar again reaches, finally a state of magnetic saturation at the point 5, which represents negative saturation. Now if the value of H is increased from negative saturation ($= 45$) to positive saturation ($= 12$) a

curve '5,6,7,2' is obtained. The closed loop "2,3,4,5,6,7,2" thus represents one complete cycle of magnetisation and is known as hysteresis loop.

CHAPTER 02

COUPLED CIRCUITS

It is defined as the interconnected loops of an electric network through the magnetic circuit.

There are two types of induced emf.

- (1) Statically Induced emf.
- (2) Dynamically Induced emf.

Faraday's Laws of Electro-Magnetic :

Introduction → First Law :→

Whenever the magnetic flux linked with a circuit changes, an emf is induced in it.

OR

Whenever a conductor cuts magnetic flux an emf is induced in it.

Second Law :→

It states that the magnitude of induced emf is equal to the rate of change of flux linkages.

OR

The emf induced is directly proportional to the rate of change of flux and number of turns

Mathematically :

$$e \propto \frac{d\phi}{dt}$$

$$e \propto N$$

Or

$$e = -N \frac{d\phi}{dt}$$

Where e = induced emf

N = No. of turns

ϕ = flux

'- ve' sign is due to Lenz's Law

Inductance :→

It is defined as the property of the substance which opposes any change in Current & flux.

Unit :→ Henry

Fleming's Right Hand Rule: —

It states that "hold your right hand with fore-finger, middle finger and thumb at right angles to each other. If the fore-finger represents the direction of field, thumb represents the direction of motion of the conductor, then the middle finger represents the direction of induced emf."

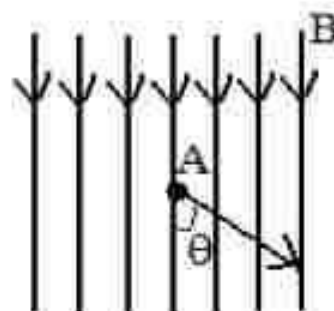
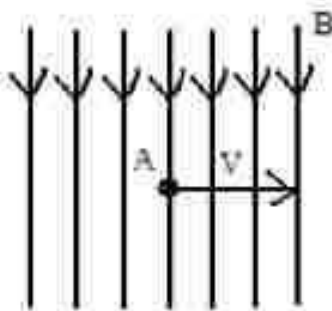
Lenz's Law : —

It states that electromagnetically induced current always flows in such a direction that the action of magnetic field set up by it tends to oppose the vary cause which produces it.

OR

It states that the direction of the induced current (emf) is such that it opposes the change of magnetic flux.

(2) Dynamically Induced emf :—



In this case the field is stationary and the conductors are rotating in an uniform magnetic field at flux density ' B ' Wb/m² and the conductor is lying perpendicular to the magnetic field. Let ' l ' is the length of the conductor and it moves a distance of ' dx ' m in time ' dt ' second.

The area swept by the conductor = $l \cdot dx$

Hence the flux cut = $l dx \cdot B$

Change in flux in time ' dt ' second =

$$\frac{Bl dx}{dt}$$

$$E = Blv$$

If the conductor is making an angle ' θ ' with the magnetic field, then

$$e = Blv \sin \theta$$

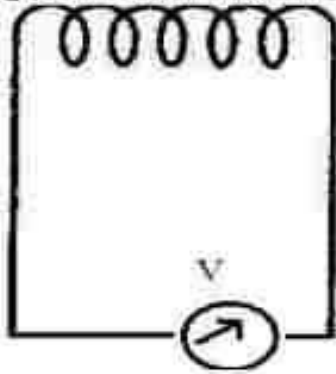
(1) Statically Induced emf :→

Here the conductors are remain in stationary and flux linked with it changes by increasing or decreasing.

It is divided into two types .

- (i) Self-induced emf.
- (ii) Mutually-induced emf.

(i) Self-induced emf :→ It is defined as the emf induced in a coil due to the change of its own flux linked with the coil.



If current through the coil is changed then the flux linked with its own turn will also change which will produce an emf is called self-induced emf.

Self-Inductance :→

It is defined as the property of the coil due to which it opposes any change (increase or decrease) of current or flux through it.

Co-efficient of Self-Inductance (L) :→

It is defined as the ratio of weber turns per ampere of current in the coil.

OR

It is the ratio of flux linked per ampere of current in the coil.

1st Method for 'L' :→

$$L = \frac{N\phi}{I}$$

Where L = Co-efficient of self-induction

N = Number of turns

ϕ = flux

I = Current

2nd Method for L :→

We know that

$$L = \frac{N\phi}{I}$$

$$\Rightarrow LI = N\phi$$

$$\Rightarrow -LI = -N\phi$$

$$\Rightarrow -L \frac{dI}{dt} = -N \frac{d\phi}{dt}$$

$$\Rightarrow -L \frac{dI}{dt} = -N \frac{d\phi}{dt}$$

$$\Rightarrow -L \frac{dI}{dt} = e_i$$

$$\Rightarrow L \frac{dI}{dt} = -e_i$$

$$\Rightarrow L = \frac{-e_i}{\frac{dI}{dt}}$$

Where L = Inductance

$e_i = -N \frac{d\phi}{dt}$ is known as self-induced emf.

When $\frac{dI}{dt} = 1 \text{ amp/sec.}$

$e = 1 \text{ volt}$

$L = 1 \text{ Henry}$

A coil is said to be a self-inductance of 1 Henry if 1 volt is induced in it. When the current through it changes at the rate of 1 amp/ sec.

3rd Method for L :→

$$L = \frac{\mu_r \mu_0 AN^2}{l}$$

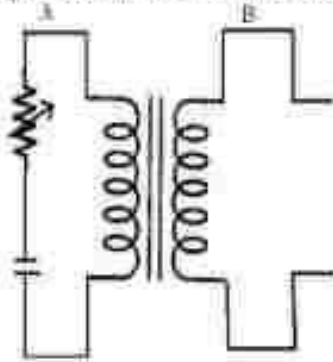
Where A = Area of x-section of the coil

N = Number of turns

l = Length of the coil

(ii) Mutually Induced emf :→

It is defined as the emf induced in one coil due to change in current in other coil. Consider two coils 'A' and 'B' lying close to each other. An emf will be induced in coil 'B' due to change of current in coil 'A' by changing the position of the rheostat.

**Mutual Inductance :→**

It is defined as the emf induced in coil 'B' due to change of current in coil 'A' is the ratio of flux linkage in coil 'B' to 1 amp. Of current in coil 'A'.

Co-efficient of Mutual Inductance (M):

Coefficient of mutual inductance between the two coils is defined as the weber-turns in one coil due to one ampere current in the other.

1st Method for 'M' :→

$$M = \frac{N_2 \phi_1}{I_1}$$

N_2 = Number of turns

M = Mutual Inductance

ϕ_1 = flux linkage

I_1 = Current in ampere

2nd Method for M :→

We know that

$$M = \frac{N_2 \phi_1}{I_1}$$

$$\Rightarrow MI_1 = N_2 \phi_1$$

$$\Rightarrow -MI_1 = N_2 \phi_1$$

$$\begin{aligned}
 &\Rightarrow -M \frac{dI_1}{dt} = -N_2 \frac{d\phi_1}{dt} \\
 &\Rightarrow -M \frac{dI_1}{dt} = e_M \\
 &\Rightarrow M \frac{dI_1}{dt} = -e_M \\
 &\Rightarrow M = \frac{-e_M}{\frac{dI_1}{dt}}
 \end{aligned}$$

Where

$$e_M = -N_2 \frac{d\phi_1}{dt}$$

$e_M = -1 \text{ VOLT}$

Then $M = 1 \text{ Henry}$

A coil is said to be a mutual inductance of 1 Henry when 1 volt is induced when the current of 1 amp/sec. is changed in its neighbouring coil.

3rd Method for M :—

$$M = \frac{M_r M_r \mu N_1 N_2}{l}$$

Co-efficient of Coupling :

Consider two magnetically coupled coils having N_1 and N_2 turns respectively. Their individual co-efficient of self-inductances are

$$L_1 = \frac{M_r M_r \mu N_1^2}{l}$$

$$L_2 = \frac{M_r M_r \mu N_2^2}{l}$$

The flux ϕ_1 produced in coil 'A' due to a current of I_1 ampere is

$$\phi_1 = \frac{L_1 I_1}{N_1} = \frac{M_r M_r \mu N_1^2}{l} \times \frac{I_1}{N_1}$$

$$\phi_1 = \frac{M_r M_r \mu N_1 I_1}{l}$$

Suppose a fraction of this flux i.e. $K_1 \phi_1$ is linked with coil 'B'

$$\text{Then } M = \frac{K_1 \phi_1}{I_1} \times N_2 = \frac{K_1 N_1 N_2}{l \mu_r \mu_r \mu} \quad (1)$$

Similarly the flux ϕ_2 produced in coil 'B' due to I_2 amp. is

$$\phi_2 = \frac{M_1 M_2 AN_2 I_1}{l}$$

Suppose a fraction of this flux i.e. $K_2 \phi_2$ is linked with coil 'A'

$$\text{Then } M = \frac{K_2 \phi_2}{I_1} \times N_2 = \frac{K_2 N_1 N_2}{l M_1 M_2 A} \quad \text{----- (2)}$$

Multiplying equation (1) & (2)

$$\begin{aligned} M^2 &= \frac{K_1 K_2 N_1^2 N_2^2}{l^2 M_1^2 M_2^2 A^2} \times N_2 \\ &= K^2 \left(\frac{M_1 M_2 AN_1^2}{l} \right) \left(\frac{M_1 M_2 AN_2^2}{l} \right) \\ &\quad [\because K_1 = K_2 = K] \end{aligned}$$

$$M^2 = K^2 L_1 L_2$$

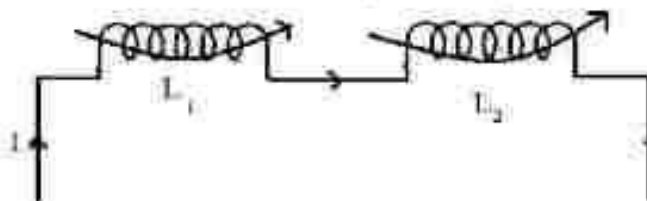
$$K^2 = \frac{M^2}{L_1 L_2}$$

$$\rightarrow K = \sqrt{\frac{M}{L_1 L_2}}$$

Where 'K' is known as the co-efficient of coupling.

Co-efficient of coupling is defined as the ratio of mutual inductance between two coils to the square root of their self- inductances.

Inductances In Series (Additive) :—



Fluxes are in the same direction

Let M = Co-efficient of mutual inductance

L_1 = Co-efficient of self-inductance of first coil.

L_2 = Co-efficient of self-inductance of second coil.

EMF induced in first coil due to self-inductance

$$e_{L_1} = -L_1 \frac{dI}{dt}$$

Mutually induced emf in first coil

$$e_{M_1} = -M \frac{dI}{dt}$$

EMF induced in second coil due to self induction

$$e_{L_2} = -L_2 \frac{dI}{dt}$$

Mutually induced emf in second coil

$$e_{M_2} = -M \frac{dI}{dt}$$

Total induced emf

$$E = e_{L_1} + e_{L_2} + e_{M_1} + e_{M_2}$$

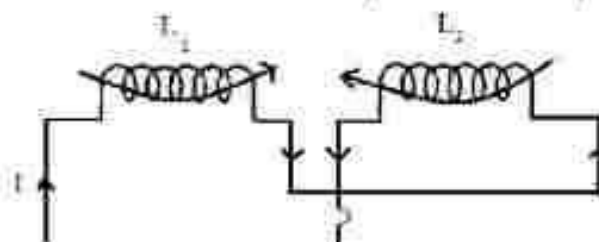
If 'L' is the equivalent inductance, then

$$-L \frac{dI}{dt} = -L_1 \frac{dI}{dt} - M \frac{dI}{dt} - L_2 \frac{dI}{dt} - M \frac{dI}{dt}$$

$$\Rightarrow -L \frac{dI}{dt} = -\frac{dI}{dt} (L_1 + L_2 + 2M)$$

$$\Rightarrow L = L_1 + L_2 + 2M$$

Inductances In Series (Subtractive) :→



(Fluxes are opposite in direction)

Let M = Co-efficient of mutual inductance

L_1 = Co-efficient of self-inductance of first coil

L_2 = Co-efficient of self-inductance of second coil

Emf induced in first coil due to self induction

$$e_{L_1} = -L_1 \frac{dI}{dt}$$

Mutually induced emf in first coil

$$e_{M_1} = -\left(-M \frac{dI}{dt}\right) = M \frac{dI}{dt}$$

Emf induced in second coil due to self-induction

$$e_{L_2} = -L_2 \frac{di_2}{dt}$$

Mutually induced emf in second coil

$$e_{M_2} = -\left(-M \frac{di_1}{dt}\right) = M \frac{di_1}{dt}$$

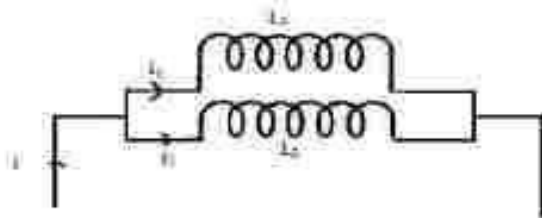
Total induced emf

$$e = e_{L_1} + e_{L_2} + e_{M_1} + e_{M_2}$$

Then

$$\begin{aligned} -L \frac{di}{dt} &= -L_1 \frac{di_1}{dt} - L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} + M \frac{di_2}{dt} \\ \Rightarrow -L \frac{di}{dt} &= -\frac{di}{dt} (L_1 + L_2 - 2M) \quad \Rightarrow L = L_1 + L_2 - 2M \end{aligned}$$

Inductances In Parallel :—



Let two inductances of L_1 & L_2 are connected in parallel

Let the co-efficient of mutual inductance between them is M .

$i = i_1 + i_2$

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt} \quad \text{----- (1)}$$

$$e = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$= L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

$$\Rightarrow L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

$$\Rightarrow (L_1 - M) \frac{di_1}{dt} = (L_2 - M) \frac{di_2}{dt}$$

$$\Rightarrow \frac{di_1}{dt} = \frac{(L_2 - M)}{(L_1 - M)} \frac{di_2}{dt} \quad \text{----- (2)}$$

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt}$$

$$= \frac{(L_2 - M)}{(L_1 - M)} \frac{di_2}{dt} + \frac{di_2}{dt}$$

$$\Rightarrow \frac{di}{dt} = \left(\frac{L_2 - M}{L_1 - M} + 1 \right) \frac{di_2}{dt} \quad \text{----- (3)}$$

If 'L' is the equivalent inductance

$$\begin{aligned} \epsilon &= L \frac{di}{dt} = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \\ L \frac{di}{dt} &= L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \\ \Rightarrow \frac{di}{dt} &= \frac{1}{L} \left[L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \right] \dots \dots \dots (4) \end{aligned}$$

Substituting the value of $\frac{di_1}{dt}$

$$\frac{di}{dt} = \frac{1}{L} \left[L_1 \frac{L_2 - M}{L_1 L_2 - M} \right] \frac{di_2}{dt} \dots \dots \dots (5)$$

Equating equation (3) & (5)

$$\begin{aligned} \left[\left(\frac{L_2 - M}{L_1 - M} \right) + 1 \right] \frac{di_2}{dt} &= \frac{1}{L} \left[L_1 \left(\frac{L_2 - M}{L_1 L_2 - M} \right) + M \right] \frac{di_2}{dt} \\ \Rightarrow \frac{L_2 - M}{L_1 - M} + 1 &= \frac{1}{L} \left[L_1 \left(\frac{L_2 - M}{L_1 L_2 - M} \right) + M \right] \\ \Rightarrow \frac{L_2 - M + L_1 - M}{L_1 - M} &= \frac{1}{L} \left[\frac{L_1 L_2 - L_1 M + L_1 M - M^2}{L_1 L_2 - M} \right] \\ \Rightarrow \frac{L_1 + L_2 - 2M}{L_1 - M} &= \frac{1}{L} \left[\frac{L_1 L_2 - M^2}{L_1 L_2 - M} \right] \\ \Rightarrow L_1 + L_2 - 2M &= \frac{1}{L} [L_1 L_2 - M^2] \end{aligned}$$

$$\Rightarrow L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

When mutual field assist.

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$$

When mutual field opposes.

Exp. -01 :

Two coupled coils have self inductances $L_1 = 10 \times 10^{-3} \text{H}$ and $L_2 = 20 \times 10^{-3} \text{H}$. The coefficient of coupling (K) being 0.75 in the air, find voltage in the second coil and the flux of first coil provided the second coils has 500 turns and the circuit current is given by $i_1 = 2 \sin 314.1t$.

Solution :

$$M = K \sqrt{L_1 L_2}$$

$$M = 0.75 \sqrt{10 \times 10^{-3} \times 20 \times 10^{-3}}$$

$$\Rightarrow M = 10.6 \times 10^{-3} \text{H}$$

The voltage induced in second coil is

$$V_2 = M \frac{di_1}{dt}$$

$$= 10.6 \times 10^{-3} \times 2 \times 314 \cos 314t$$

The magnetic circuit being linear,

$$M = \frac{N_1 \phi_1}{i_1} = \frac{500 \times (K \phi_1)}{i_1}$$

$$\phi = \frac{M}{500 \times K} \times i_1 = \frac{10.6 \times 10^{-3}}{500 \times 0.75} \times 2 \sin 314t$$

$$= 5.66 \times 10^{-5} \sin 314t$$

Exp. 02

Find the total inductance of the three series connected coupled coils. Where the self and mutual inductances are

$$L_1 = 1\text{H}, L_2 = 2\text{H}, L_3 = 5\text{H}$$

$$M_{12} = 0.5\text{H}, M_{23} = 1\text{H}, M_{13} = 1\text{H}$$

Solution:

$$L_A = L_1 + M_{12} + M_{13}$$

$$= 1 + 0.5 + 1$$

$$= 2.5\text{H}$$

$$L_B = L_2 + M_{23} + M_{12}$$

$$= 2 + 1 + 0.5$$

$$= 3.5\text{H}$$

$$L_C = L_3 + M_{23} + M_{13}$$

$$= 5 + 1 + 1$$

$$= 7\text{H}$$

Total inductances are

$$\begin{aligned}L_{eq} &= L_A + L_B + L_C \\&= 2.5 + 3.5 + 7 \\&= 13H \text{ (Ans)}\end{aligned}$$

CHAPTER 3

Circuit Elements and analysis

1.1 Voltage

Energy is required for the movement of charge from one point to another. Let W Joules of energy be required to move positive charge Q coulombs from a point a to point b in a circuit. We say that a voltage exists between the two points. The voltage V between two points may be defined in terms of energy that would be required if a charge were transferred from one point to the other. Thus, there can be a voltage between two points even if no charge is actually moving from one to the other. Voltage between a and b is given by

$$V = \frac{W}{Q} \text{ J/C}$$

$$\text{Hence Electric Potential (V)} = \frac{\text{Worked are (W) in Joules}}{\text{Charge (Q) in coulombs}}$$

Current:

An electric current is the movement of electric charges along a definite path. In case of a conductor the moving charges are electrons.

The unit of current is the ampere. The ampere is defined as that current which when flowing in two infinitely long parallel conductors of negligible cross section, situated 1 meter apart in Vacuum, produces between the conductors a force of 2×10^{-7} Newton per metre length.

Power: Power is defined as the work done per unit time. If a field F newton acts for t seconds through a distance d metres along a straight line, work done $W = F \times d$ N.m. or J. The power P , either generated or dissipated by the circuit element.

$$P = \frac{W}{t} = \frac{F \times d}{t}$$

Power can also be written as $\text{Power} = \frac{\text{Work}}{\text{time}}$

$$= \frac{\text{Work}}{\text{Charge}} \times \frac{\text{Charge}}{\text{Time}} = \text{Voltage} \times \text{Current}$$

$$P = V \times I \text{ watt.}$$

Energy : Electric energy W is defined as the Power Consumed in a given time. Hence, if current I A flows in an element over a time period t second, when a voltage V volts is applied across it, the energy consumed is given by

$$W = P \times t = V \times I \times t \text{ J or watt. second.}$$

The unit of energy W is Joule (J) or watt. second. However, in practice, the unit of energy is kilowatt. hour (Kwh)

1.2 **Resistance** : According to Ohm's law potential difference (V) across the ends of a conductor is proportional to the current (I) flowing through the conductor at a constant temperature. Mathematically Ohm's law is expressed as

$$V \propto I \text{ or } V = R \times I$$

Or $R = \frac{V}{I}$ where R is the proportionality constant and is designated as the conductor resistance and has the unit of Ohm (Ω).

Conductance : Voltage is induced in a stationary conductor when placed in a varying magnetic field. The induced voltage (e) is proportional to the time rate of change of current, di/dt producing the magnetic field.

$$\text{Therefore } e \propto \frac{di}{dt}$$

$$\text{Or } e = L \frac{di}{dt}$$

e and i are both function of time. The proportionality constant L is called inductance. The Unit of inductance is Henry (H).

Capacitance : A capacitor is a Physical device, which when polarized by an electric field by applying a suitable voltage across it, stores energy in the form of a charge separation.

The ability of the capacitor to store charge is measured in terms of capacitance. Capacitance of a capacitor is defined as the charge stored per Volt applied.

$$C = \frac{q}{v} = \frac{\text{Coulomb}}{\text{Volt}} = \text{Farad}$$

1.3 **Active and passive Branch :**

A branch is said to be active when it contains one or more energy sources. A passive branch does not contain an energy source.

Branch : A branch is an element of the network having only two terminals.

Bilateral and unilateral element :

A bilateral element conducts equally well in either direction. Resistors and inductors are examples of bilateral elements. When the current voltage relations are different for the two directions of current flow, the element is said to be unilateral. Diode is an unilateral element.

Linear Elements : When the current and voltage relationship in an element can be simulated by a linear equation either algebraic, differential or integral type, the element is said to be linear element.

Non Linear Elements : When the current and voltage relationship in an element can not be simulated by a linear equation, the element is said to be non linear elements.

1.4 **Kirchhoff's Voltage Law (KVL) :**

The algebraic sum of Voltages (or voltage drops) in any closed path or loop is Zero.

Application of KVL with series connected voltage source.

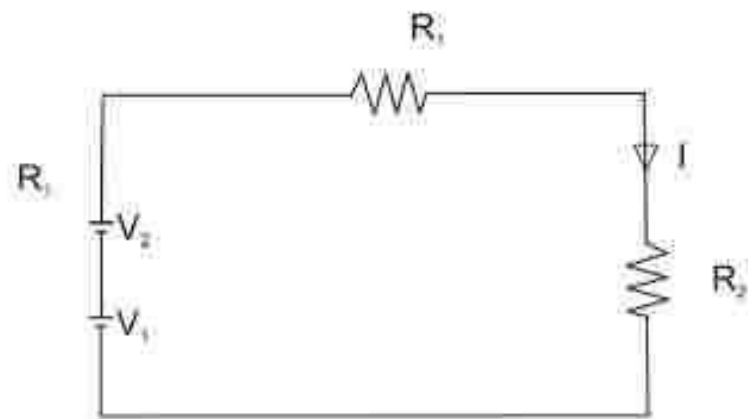


Fig. 1.1

$$V_1 + V_2 - IR_1 - IR_2 = 0$$

$$\Rightarrow V_1 + V_2 = I (R_1 + R_2)$$

$$I = \frac{V_1 + V_2}{R_1 + R_2}$$

Application of KVL while voltage sources are connected in opposite polarity.

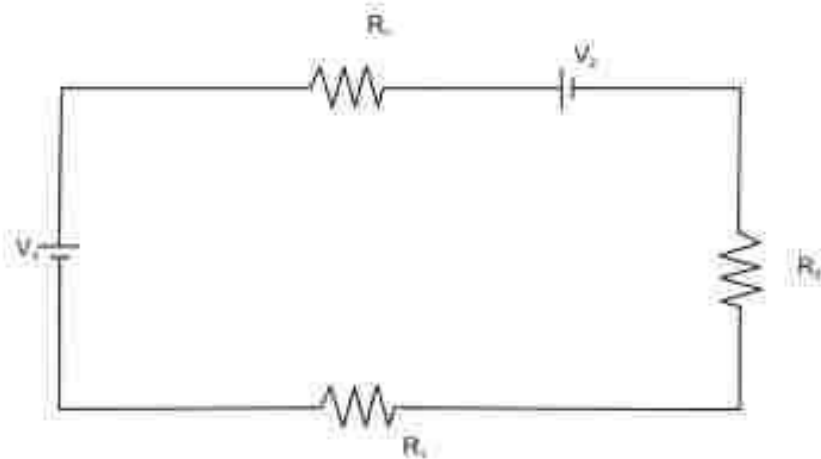


Fig. 1.2

$$V_1 - IR_1 - V_2 - IR_2 - IR_3 = 0$$

$$\Rightarrow V_1 - V_2 = IR_1 + IR_2 + IR_3$$

$$\Rightarrow V_1 - V_2 = I (R_1 + IR_2 + IR_3)$$

$$\Rightarrow I = \frac{V_1 - V_2}{R_1 + R_2 + R_3}$$

Kirchoff's Current Law (KCL) :

The algebraic sum of currents meeting at a junction or node is zero.

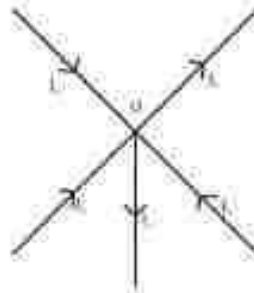


Fig. 1.3

Considering five conductors, carrying currents I_1 , I_2 , I_3 , I_4 and I_5 meeting at a point O. Assuming the incoming currents to be positive and outgoing currents negative,

$$I_1 + (-I_2) + I_3 + (-I_4) + I_5 = 0$$

$$I_1 - I_2 + I_3 - I_4 + I_5 = 0$$

$$I_1 + I_3 + I_5 = I_2 + I_4$$

Thus above Law can also be stated as the sum of currents flowing towards any junction in an electric circuit is equal to the sum of the currents flowing away from that junction.

Voltage Division (Series Circuit)

Considering a voltage source (E) with resistors R_1 and R_2 in series across it.

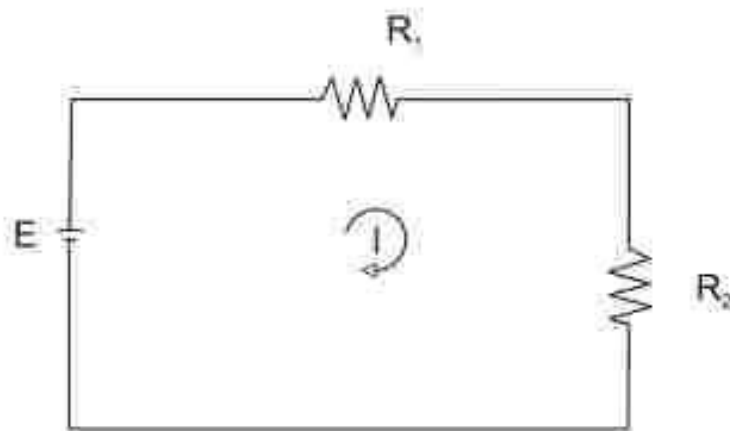


Fig. 1.4

$$I = \frac{E}{R_1 + R_2}$$

$$\text{Voltage drop across } R_1 = I \cdot R_1 = \frac{E \cdot R_1}{R_1 + R_2}$$

$$\text{Similarly voltage drop across } R_2 = I \cdot R_2 = \frac{E \cdot R_2}{R_1 + R_2}$$

Current Division :

A parallel circuit acts as a current divider as the current divides in all branches in a parallel circuit.

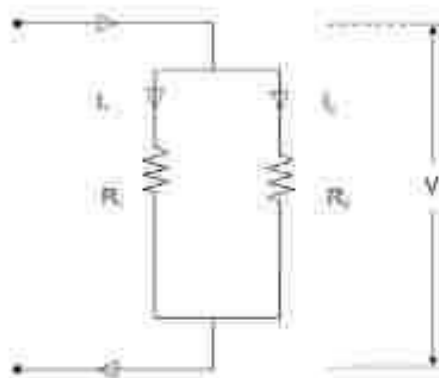


Fig. 1.5

Fig. shown the current \$I\$ has been divided into \$I_1\$ and \$I_2\$ in two parallel branches with resistances \$R_1\$ and \$R_2\$ while \$V\$ is the voltage drop across \$R_1\$ and \$R_2\$.

$$I_1 = \frac{V}{R_1} \text{ and } I_2 = \frac{V}{R_2}$$

Let \$R\$ = Total resistance of the circuit.

$$\text{Hence } \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$\Rightarrow R = \frac{R_1 R_2}{R_1 + R_2}$$

$$I = \frac{V}{R} = \frac{V}{\frac{R_1 R_2}{R_1 + R_2}} = \frac{V(R_1 + R_2)}{R_1 R_2}$$

$$\text{But } V = I_1 R_1 = I_2 R_2$$

$$\Rightarrow I = I_1 R_1 \left(\frac{R_1 + R_2}{R_1 R_2} \right)$$

$$\Rightarrow I = \frac{I_1 (R_1 + R_2)}{R_2}$$

Therefore
$$I_1 = \frac{IR_2}{R_1 + R_2}$$

Similarly it can be derived that

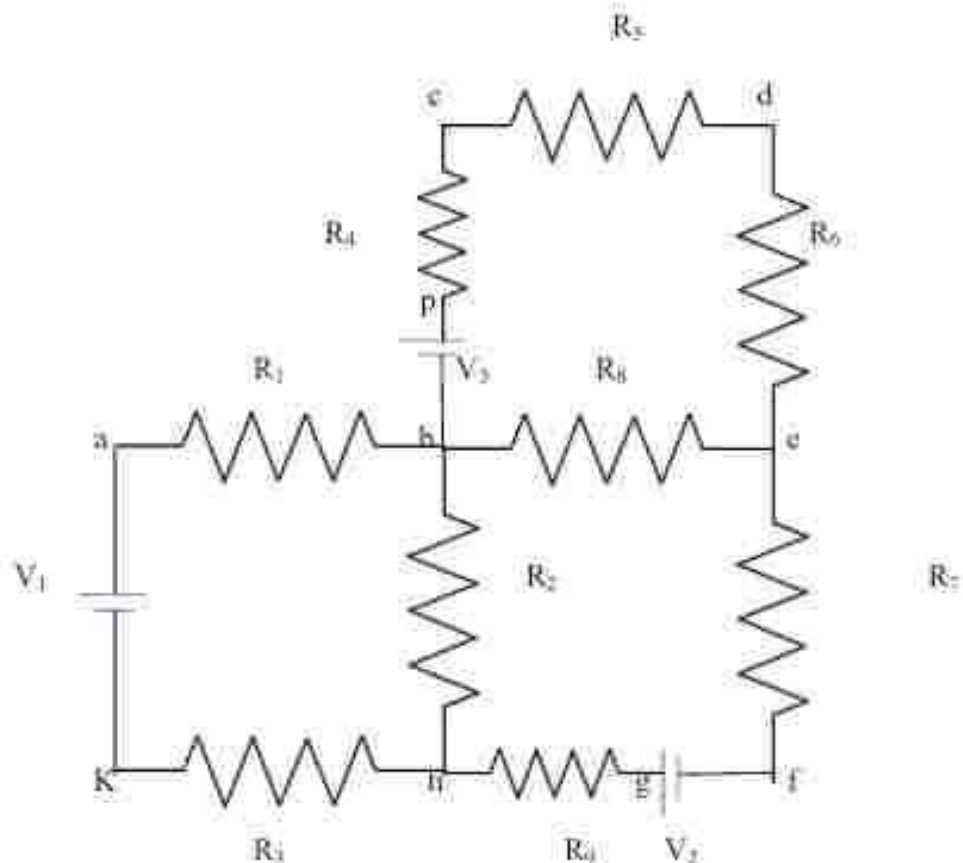
$$I_2 = \frac{IR_1}{R_1 + R_2}$$

NETWORK ANALYSIS

Different terms are defined below:

1. **Circuit:** A circuit is a closed conducting path through which an electric current either flow or is intended flow
2. **Network:** A combination of various electric elements, connected in any manner. Whatsoever, is called an electric network
3. **Node:** it is an equipotential point at which two or more circuit elements are joined.
4. **Junction:** it is that point of a network where three or more circuit elements are joined.
5. **Branch:** it is a part of a network which lies between junction points.
6. **Loop:** It is a closed path in a circuit in which no element or node is accounted more than once.
7. **Mesh:** It is a loop that contains no other loop within it.

Example 3.1 In this circuit configuration of figure 3.1, obtain the no. of i) circuit elements ii) nodes iii) junction points iv) branches and v) meshes.



Solution: i) no. of circuit elements = 12 (9 resistors + 3 voltage sources)

ii) no. of nodes = 10 (a, b, c, d, e, f, g, h, k, p)

iii) no. of junction points = 3 (b, e, h)

iv) no. of branches = 5 (bcde, be, bh, befh, bakh)

v) no. of meshes = 3 (abhk, bcde, befh)

3.2 MESH ANALYSIS

Mesh and nodal analysis are two basic important techniques used in finding solutions for a network. The suitability of either mesh or nodal analysis to a particular problem depends mainly on the number of voltage sources or current sources. If a network has a large number of voltage sources, it is useful to use mesh analysis; as this analysis requires that all the sources in a circuit be voltage sources. Therefore, if there are any current sources in a circuit they are to be converted into equivalent voltage sources, if, on the other hand, the network has more current sources, nodal analysis is more useful.

Mesh analysis is applicable only for planar networks. For non-planar circuits mesh analysis is not applicable. A circuit is said to be planar, if it can be drawn on a plane surface without crossovers. A non-planar circuit cannot be drawn on a plane surface without a crossover.

Figure 3.2 (a) is a planar circuit. Figure 3.2 (b) is a non-planar circuit and fig. 3.2 (c) is a planar circuit which looks like a non-planar circuit. It has already been discussed that a loop is a closed path. A mesh is defined as a loop which does not contain any other loops within it. To apply mesh analysis, our first step is to check whether the circuit is planar or not and the second is to select mesh currents. Finally, writing Kirchhoff's voltage law equations in terms of unknowns and solving them leads to the final solution.

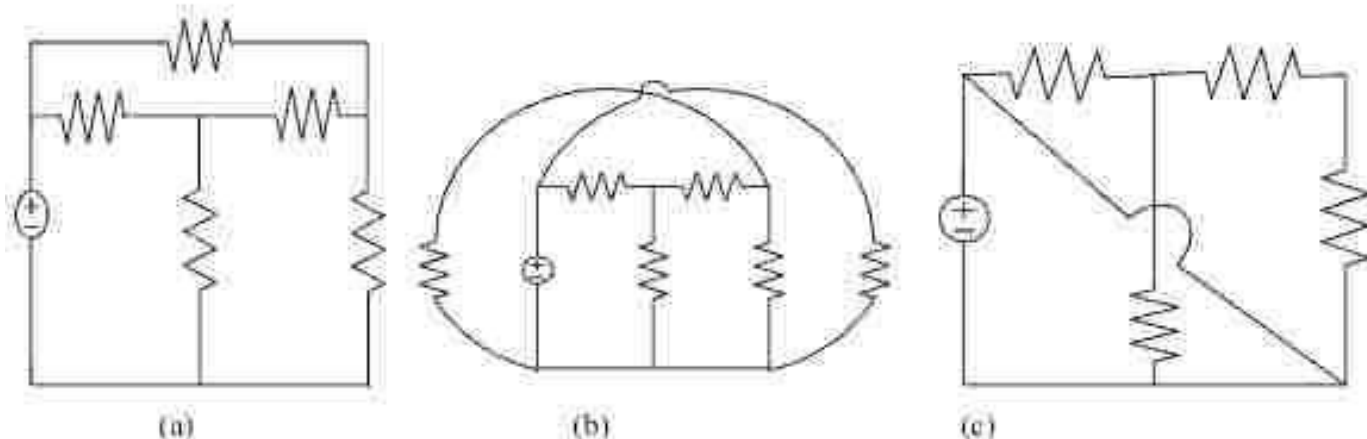


Figure 3.2

Observation of the Fig.3.2 indicates that there are two loops abefa and bedeb in the network. Let us assume loop currents I_1 and I_2 with directions as indicated in the figure.

Considering the loop abefa alone, we observe that current I_1 is passing through R_1 and $(I_1 - I_2)$ is passing through R_2 . By applying Kirchhoff's voltage law, we can write

$$V_s = I_1 R_1 + R_2 (I_1 - I_2) \quad (3.1)$$

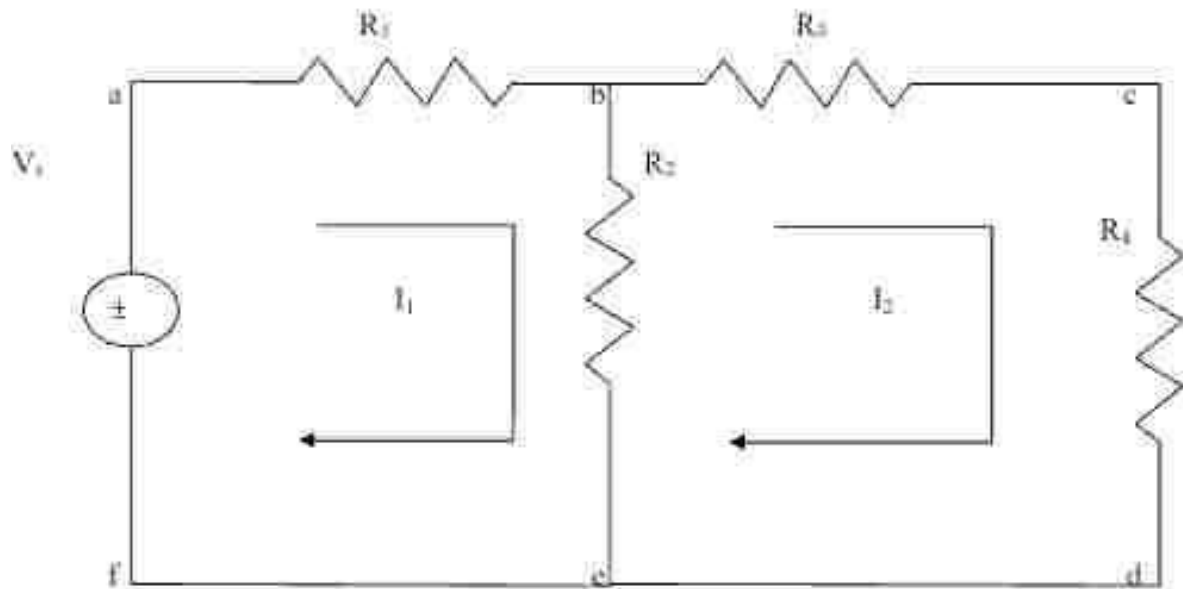


Figure 3.3

Similarly, if we consider the second mesh bedeb, the current I_2 is passing through R_3 and R_4 and $(I_2 - I_1)$ is passing through R_2 . By applying Kirchhoff's voltage law around the second mesh, we have

$$R_2 (I_2 - I_1) + R_3 I_2 + R_4 I_2 = 0 \quad (3.2)$$

By rearranging the above equations, the corresponding mesh current equations are

$$\begin{aligned} I_1 (R_1 + R_2) - I_2 R_2 &= V_s \\ -I_1 R_2 + (R_2 + R_3 + R_4) I_2 &= 0 \end{aligned} \quad (3.3)$$

By solving the above equations, we can find the currents I_1 and I_2 . If we observe Fig. 3.3, the circuit consists of five branches and four nodes, including the reference node. The number of mesh currents is equal to the number of mesh equations.

And the number of equations = branches - (nodes - 1). In Fig. 3.3, the required number of mesh current would be $5 - (4 - 1) = 2$.

In general we have B number of branches and N number of nodes including the reference node than number of linearly independent mesh equations $M=B-(N-1)$.

Example 3.2 Write the mesh

current equations in the circuit shown

in fig 3.4 and determine the currents.

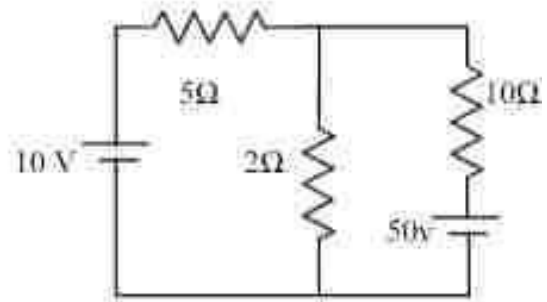


Figure 3.4

Solution: Assume two mesh currents in the direction as indicated in fig 3.5. The mesh current equations are

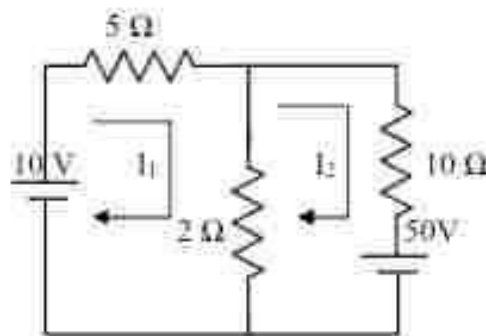


Figure 3.5

$$5I_1 + 2(I_1 - I_2) = 10$$

$$10I_2 + 2(I_2 - I_1) + 50 = 0 \quad (3.4)$$

We can rearrange the above equations as

$$7I_1 - 2I_2 = 10$$

$$-2I_1 + 12I_2 = -50 \quad (3.5)$$

By solving the above equations, we have $I_1 = 0.25$ A, and $I_2 = -4.125$

Here the current in the second mesh I_2 is negative; that is the actual current I_2 flows opposite to the assumed direction of current in the circuit of fig.3.3.

Example 3.3 Determine the mesh current I_1 in the circuit shown in fig.3.6.

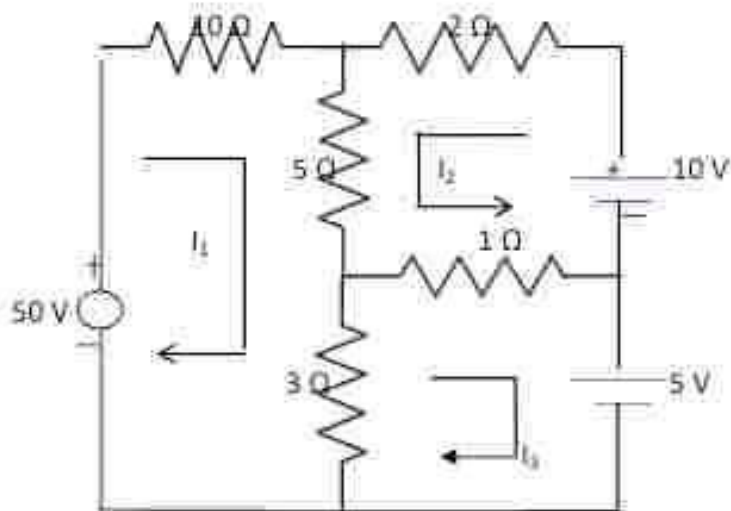


Figure 3.6

Solution: From the circuit, we can form the following three mesh equations

$$10I_1 + 5(I_1 + I_2) + 3(I_1 - I_3) = 50 \quad (3.6)$$

$$2I_2 + 5(I_2 + I_1) + 1(I_2 + I_3) = 10 \quad (3.7)$$

$$3(I_2 - I_1) + 1(I_3 + I_2) = -5 \quad (3.8)$$

Rearranging the above equations we get

$$18I_1 + 5I_2 - 3I_3 = 50 \quad (3.9)$$

$$5I_1 + 8I_2 + I_3 = 10 \quad (3.10)$$

$$-3I_1 + I_2 + 4I_3 = -5 \quad (3.11)$$

According to the Cramer's rule

$$I_1 = \frac{\begin{vmatrix} 50 & 5 & -3 \\ 10 & 8 & 1 \\ -5 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{vmatrix}} = \frac{1175}{356}$$

Or $I_1 = 3.3 \text{ A}$ Similarly,

$$I_2 = \frac{\begin{vmatrix} 18 & 50 & -3 \\ 5 & 10 & 1 \\ -3 & -5 & 4 \end{vmatrix}}{\begin{vmatrix} 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{vmatrix}} = \frac{-355}{356}$$

Or $I_2 = -0.997 \text{ A}$

(3.12)

$$I_3 = \frac{\begin{vmatrix} 18 & 5 & 50 \\ 5 & 8 & 10 \\ -3 & 1 & -5 \end{vmatrix}}{\begin{vmatrix} 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{vmatrix}} = \frac{525}{356}$$

Or $I_3 = 1.47 \text{ A}$

(3.13)

$\therefore I_1 = 3.3 \text{ A}, I_2 = -0.997 \text{ A}, I_3 = 1.47 \text{ A}$

3.3 MESH EQUATIONS BY INSPECTION METHOD The mesh equations for a general planar network can be written by inspection without going through the detailed steps. Consider a three mesh networks as shown in figure 3.7.

The loop equations are $I_1 R_1 + R_2(I_1 - I_2) = V_1$

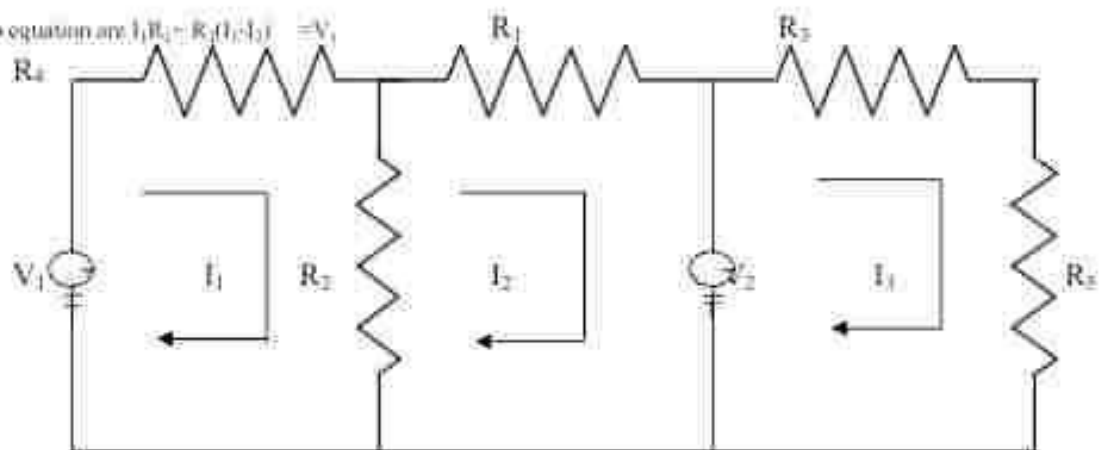


Figure 3.7

$$R_2(I_2 - I_1) + I_2 R_3 = -V_2 \quad 3.14$$

$$R_4 I_1 + R_5 I_2 = V_2 \quad 3.15$$

Reordering the above equations, we have

$$(R_1 + R_2)I_1 - R_2 I_2 = V_1 \quad 3.16$$

$$-R_2 I_1 + (R_2 + R_3)I_2 = -V_2 \quad 3.17$$

$$(R_4 + R_5)I_2 = V_2 \quad 3.18$$

The general mesh equations for three mesh resistive network can be written as

$$R_{11}I_1 \pm R_{12}I_2 \pm R_{13}I_3 = V_1 \quad 3.19$$

$$\pm R_{21}I_1 + R_{22}I_2 \pm R_{23}I_3 = V_2 \quad 3.20$$

$$\pm R_{31}I_1 \pm R_{32}I_2 + R_{33}I_3 = V_3 \quad 3.21$$

By comparing the equations 3.16, 3.17 and 3.18 with equations 3.19, 3.20 and 3.21 respectively, the following observations can be taken into account.

1. The self-resistance in each mesh
2. The mutual resistances between all pairs of meshes and
3. The algebraic sum of the voltages in each mesh.

The self-resistance of loop 1, $R_{11} = R_1 + R_2$, is the sum of the resistances through which I_1 passes.

The mutual resistance of loop 1, $R_{12} = -R_2$, is the sum of the resistances common to loop currents I_1 and I_2 . If the directions of the currents passing through the common resistances are the same, the mutual resistance will have a positive sign; and if the directions of the currents passing through the common resistance are opposite then the mutual resistance will have a negative sign.

$V_1 = V_1$ is the voltage which drives the loop 1. Here the positive sign is used if the direction of the currents is the same as the direction of the source. If the current direction is opposite to the direction of the source, then the negative sign is used.

Similarly $R_{22} = R_2 + R_3$ and $R_{33} = R_4 + R_5$ are the self-resistances of loops 2 and 3 respectively. The mutual resistances $R_{13} = 0$, $R_{31} = -R_2$, $R_{23} = 0$, $R_{32} = 0$, $R_{32} = 0$ are the sums of the resistances common to the mesh currents indicated in their subscripts.

$V_1 = -V_1$, $V_2 = V_2$ are the sum of the voltages driving their respective loops.

Example 3.4 write the mesh equation for the circuit shown in fig. 3.8

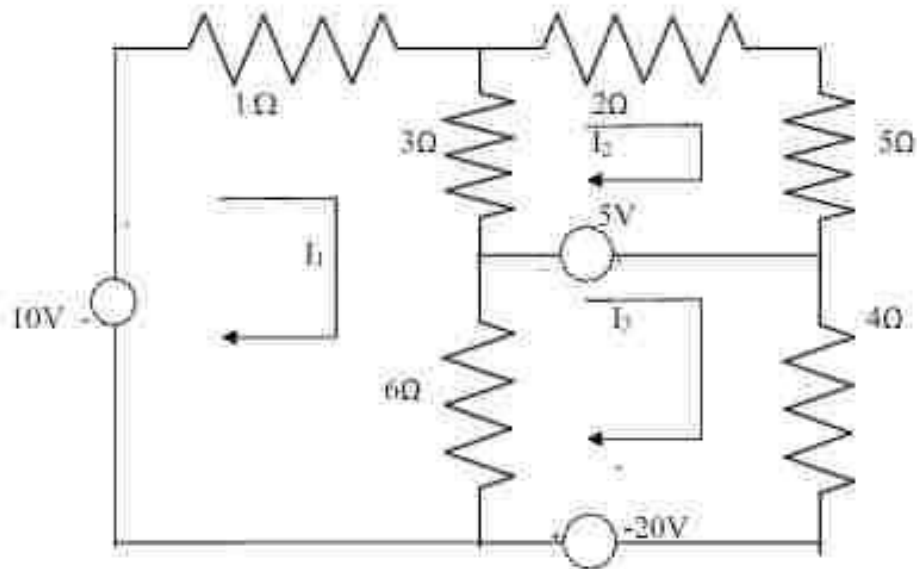


Figure 3.8

Solution : the general equation for three mesh equation are

$$R_{11}I_1 \pm R_{12}I_2 \pm R_{13}I_3 = V_s \quad (3.22)$$

$$\pm R_{21}I_1 + R_{22}I_2 \pm R_{23}I_3 = V_s \quad (3.23)$$

$$\pm R_{31}I_1 \pm R_{32}I_2 + R_{33}I_3 = V_s \quad (3.24)$$

Consider equation 3.22

R_{11} = self resistance of loop 1 = $(1\Omega + 3\Omega + 6\Omega) = 10\Omega$

R_{12} = the mutual resistance common to loop 1 and loop 2 = -3Ω

Here the negative sign indicates that the currents are in opposite direction.

R_{13} = the mutual resistance common to loop 1 & 3 = -6Ω

$V_s = +10\text{ V}$, the voltage the driving the loop 1.

Here the positive sign indicates the loop current I_1 is in the same direction as the source element.

Therefore equation 3.22 can be written as

$$10 I_1 - 3 I_2 - 6 I_3 = 10 \text{ V} \quad (3.25)$$

Consider Eq. 3.23

R_{12} = the mutual resistance common to loop 1 and loop 2 = -3Ω

R_{22} = self resistance of loop 2 = $(3 \Omega + 2 \Omega + 5 \Omega) = 10 \Omega$

$R_{23} = 0$, there is no common resistance between loop 2 and 3.

$V_2 = -5 \text{ V}$, the voltage driving the loop 2.

Therefore Eq. 3.23 can be written as

$$-3 I_1 + 10 I_2 = -5 \text{ V} \quad (3.26)$$

Consider Eq. 3.24

R_{13} = the mutual resistance common to loop 1 and loop 3 = -6Ω

R_{23} = the mutual resistance common to loop 3 and loop 2 = 0

R_{33} = self resistance of loop 3 = $(6 \Omega + 4 \Omega) = 10 \Omega$

V_3 = the algebraic sum of the voltage driving loop 3

$$= (5 \text{ V} + 20 \text{ V}) = 25 \text{ V} \quad (3.27)$$

Therefore, Eq. 3.24 can be written as $-6 I_1 + 10 I_3 = 25 \text{ V}$

$$-6 I_1 - 3 I_2 - 6 I_3 = 10 \text{ V}$$

$$-3 I_1 + 10 I_2 = -5 \text{ V}$$

$$-6 I_1 + 10 I_3 = 25 \text{ V}$$

3.4 SUPERMESH ANALYSIS

Suppose any of the branches in the network has a current source, then it is slightly difficult to apply mesh analysis straight forward because first we should assume an unknown voltage across the current source, writing mesh equation as before, and then relate the source current to the assigned mesh currents. This is generally a difficult approach. One way to overcome this difficulty is by applying the supermesh technique. Here we have to choose the kind of supermesh. A supermesh is constituted by two adjacent loops that have a common current source. As an example, consider the network shown in the figure 3.9.

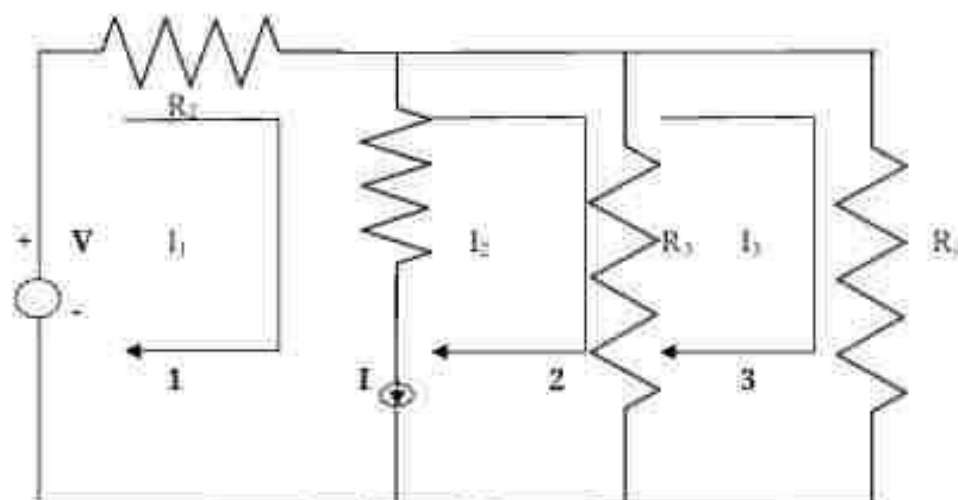


Figure 3.9

Here the current source I is in the common boundary for the two meshes 1 and 2. This current source creates a supermesh, which is nothing but a combination of meshes 1 and 2.

$$R_1 I_1 + R_3(I_2 - I_1) = V$$

$$\text{Or } R_1 I_1 + R_3 I_2 - R_3 I_1 = V$$

Considering mesh 3, we have

$$R_3(I_1 - I_2) + R_4 I_3 = 0$$

Finally, the current I from current source is equal to the difference between two mesh currents i.e.

$$I_1 - I_2 = I$$

we have thus formed three mesh equations which we can solve for the three unknown currents in the network.

Example 3.5. Determine the current in the 5Ω resistor in the network given in Fig. 3.10

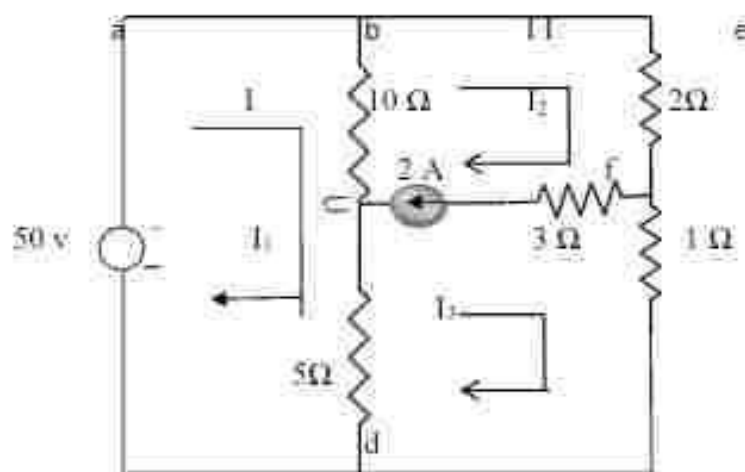


Figure 3.10

Solution: - From the first mesh, i.e. abceda, we have

$$50 = 10(I_1 - I_2) + 5(I_1 - I_3)$$

$$\text{Or } 15I_1 - 10I_2 - 5I_3 = 50 \quad (3.28)$$

From the second and third meshes, we can form a super mesh

$$10(I_2 - I_1) + 2I_2 + I_3 + 5(I_3 - I_1) = 0$$

$$\text{Or } -15I_1 + 12I_2 + 6I_3 = 0 \quad (3.29)$$

The current source is equal to the difference between II and III mesh currents

$$\text{i.e. } I_2 - I_3 = 2\text{A} \quad (3.30)$$

Solving 3.28, 3.29 and 3.30, we have

$$I_1 = 19.99\text{A}, I_2 = 17.33\text{A}, \text{ and } I_3 = 15.33\text{A}$$

The current in the 5Ω resistor $= I_1 - I_3$

$$= 19.99 - 15.33 = 4.66\text{A}$$

The current in the 5Ω resistor is 4.66A.

Example 3.6. Write the mesh equations for the circuit shown in fig. 3.11 and determine the currents, I_1 , I_2 and I_3 .

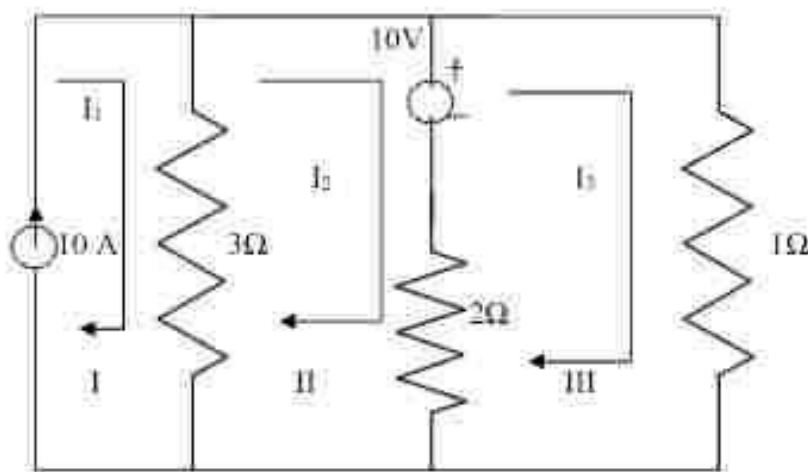


Figure 3.11

Solution ; In fig 3.11, the current source lies on the perimeter of the circuit, and the first mesh is ignored. Kirchhoff's voltage law is applied only for second and third meshes .

From the second mesh, we have

$$3(I_2 - I_1) + 2(I_2 - I_3) + 10 = 0$$

$$\text{Or } -3I_1 + 5I_2 - 2I_3 = -10 \quad (3.31)$$

From the third mesh, we have

$$I_3 + 2(I_3 - I_2) = 10$$

$$\text{Or } -2I_2 + 3I_3 = 10 \quad (3.32)$$

From the first mesh, $I_1 = 10\text{A}$ (3.33)

From the above three equations, we get

$$I_1 = 10\text{A}, \quad I_2 = 7.27, \quad I_3 = 8.18\text{A}$$

3.5 NODAL ANALYSIS

In the chapter 1 we discussed simple circuits containing only two nodes, including the reference node. In general, in a N node circuit, one of the nodes is chosen as the reference or datum node, then it is possible to write $N-1$ nodal equations by assuming $N-1$ node voltages. For example a 10 node circuit requires nine unknown voltages and nine equations. Each node in a circuit can be assigned a number or a letter. The node voltage is the voltage of a given node with respect to one particular node, called the reference node, which we assume at zero potential. In the circuit shown in fig. 3.12, node 3 is assumed as the Reference node. The voltage at node 1 is the voltage at that node with respect to node 3. Similarly, the voltage at node 2 is the voltage at that node with respect to node 3. Applying Kirchhoff's current law at node 1, the current entering is the current leaving. (See Fig 3.13)

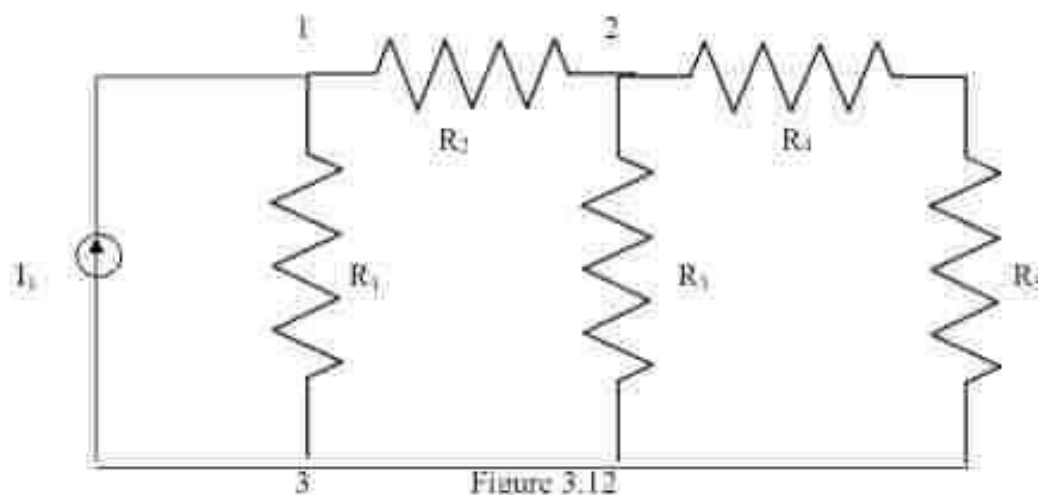


Figure 3.12

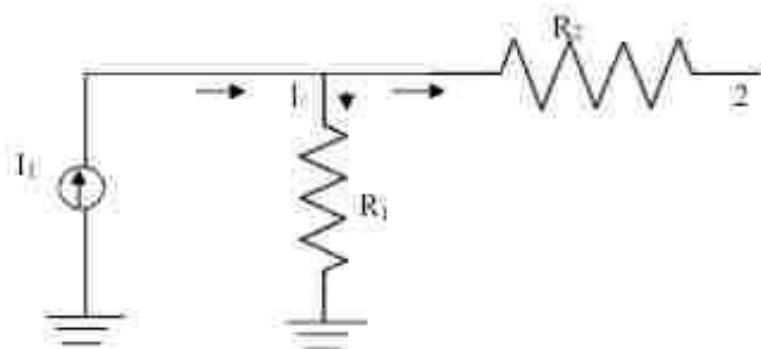


Figure 3.13

$$I_1 = V_1/R_1 + (V_1 - V_2)/R_2$$

Where V_1 and V_2 are the voltages at node 1 and 2, respectively. Similarly, at node 2, the current entering is equal to the current leaving as shown in fig. 3.14

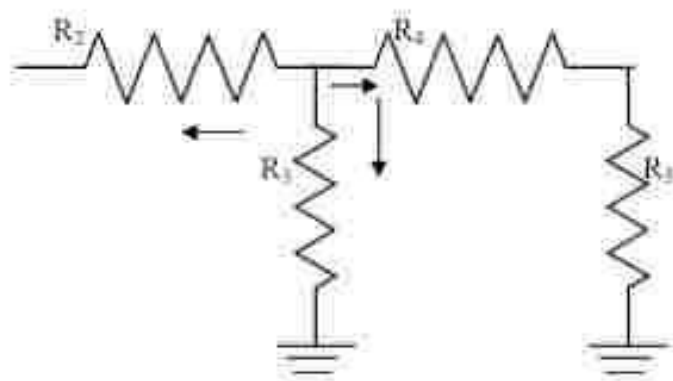


Figure 3.14

$$(V_2 - V_1)/R_2 + V_2/R_4 + V_2/(R_4 + R_5) = 0$$

Rearranging the above equations, we have

$$V_1[1/R_1 + 1/R_2] - V_2(1/R_2) = I_1$$

$$-V_1(1/R_2) + V_2[1/R_2 + 1/R_4 + 1/(R_4 + R_5)] = 0$$

From the above equations we can find the voltages at each node.

Example 3.7 Determine the voltages at each node for the circuit shown in fig 3.15

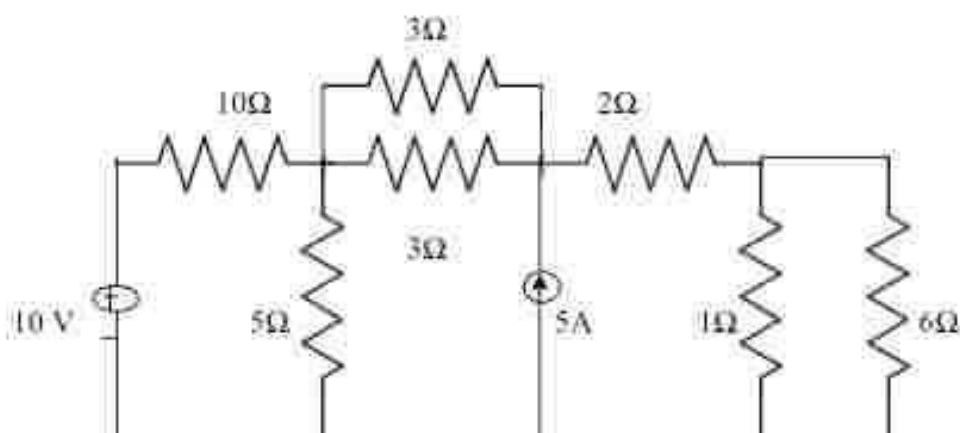


Figure 3.15

Solution : At node 1, assuming that all currents are leaving, we have

$$(V_1 - 10)/10 + (V_1 - V_2)/3 + V_1/5 + (V_1 - V_2)/3 = 0$$

$$\text{Or } V_1[1/10 + 1/3 + 1/5 + 1/3] - V_2[1/3 + 1/3] = 1$$

$$0.96V_1 - 0.66V_2 = 1$$

(3.36)

At node 2, assuming that all currents are leaving except the current from current source, we have

$$(V_2 - V_1)/3 + (V_2 - V_1)/3 + (V_2 - V_1)/2 = 5$$

$$-V_1[2/3] + V_2[1/3 + 1/3 + 1/2] - V_3(1/2) = 5$$

$$-0.66V_1 + 1.16V_2 - 0.5V_3 = 5$$

(3.37)

At node 3 assuming all currents are leaving, we have

$$(V_3 - V_2)/2 + V_3/1 + V_3/6 = 0$$

$$-0.5V_2 + 1.66V_3 = 0 \quad (3.38)$$

Applying Cramer's rule we get

$$V_1 = \frac{\begin{vmatrix} 1 & -0.66 & 0 \\ 5 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{vmatrix}}{\begin{vmatrix} 0.96 & -0.66 & 0 \\ -0.66 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{vmatrix}} = \frac{7.154}{0.887} = 8.06$$

Similarly,

$$V_2 = \frac{\begin{vmatrix} 0.96 & 1 & 0 \\ -0.66 & 5 & -0.5 \\ 0 & 0 & 1.66 \end{vmatrix}}{\begin{vmatrix} 0.96 & -0.66 & 0 \\ -0.66 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{vmatrix}} = \frac{9.06}{0.887} = 10.2$$

$$V_3 = \frac{\begin{vmatrix} 0.96 & -0.66 & 1 \\ -0.66 & 1.16 & 5 \\ 0 & -0.5 & 0 \end{vmatrix}}{\begin{vmatrix} 0.96 & -0.66 & 0 \\ -0.66 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{vmatrix}} = \frac{2.73}{0.887} = 3.07$$

3.6 NODAL EQUATIONS BY INSPECTION METHOD The nodal equations for a general planar network can also be written by inspection without going through the detailed steps. Consider a three node resistive network, including the reference node, as shown in fig 3.16

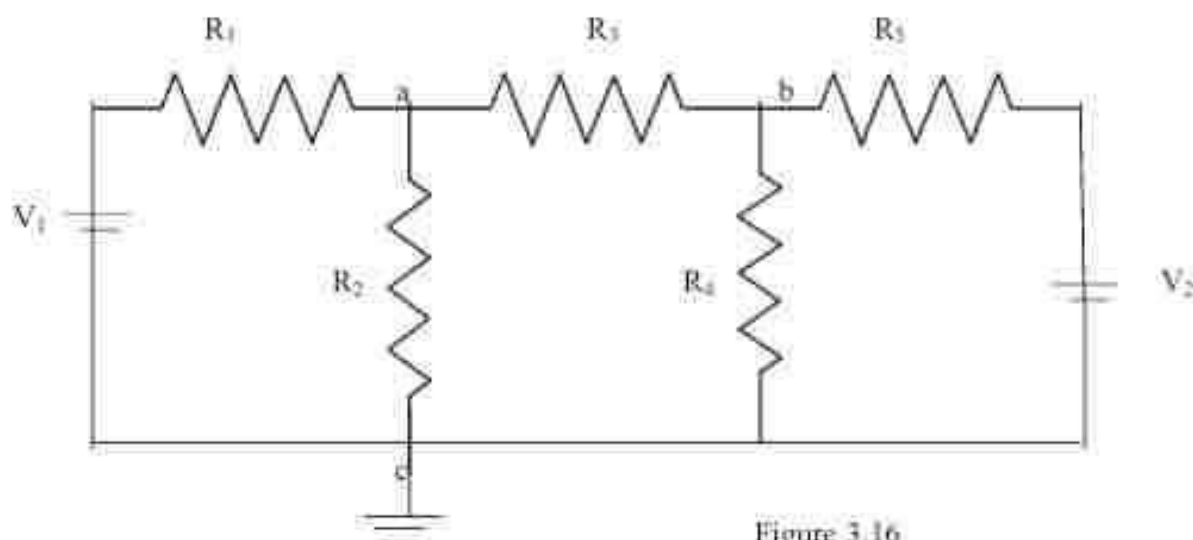


Figure 3.16

In fig. 3.16 the points a and b are the actual nodes and c is the reference node.

Now consider the nodes a and b separately as shown in fig 3.17(a) and (b)

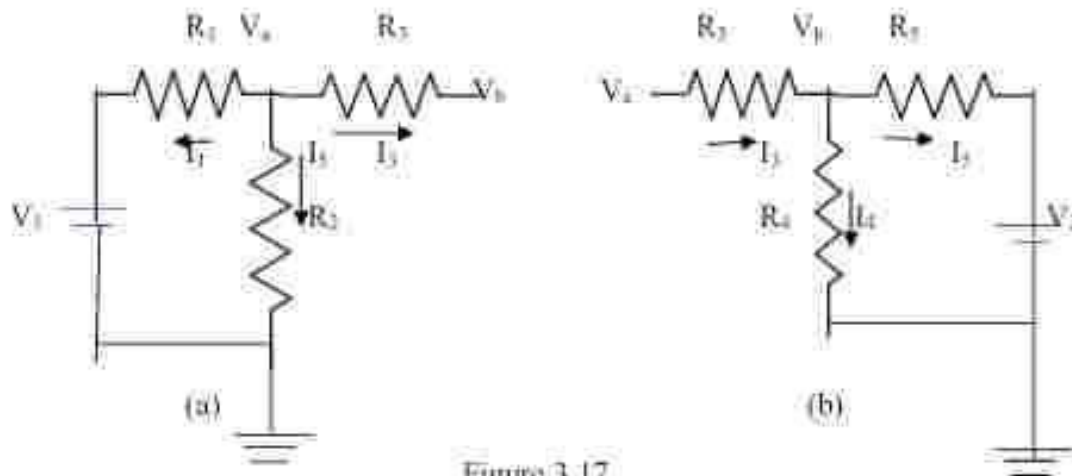


Figure 3.17

In fig 3.17 (a), according to Kirchhoff's current law we have

$$I_1 + I_2 + I_3 = 0$$

$$(V_a - V_1)/R_1 + V_a/R_2 + (V_a - V_b)/R_3 = 0 \quad (3.39)$$

In fig 3.17 (b), if we apply Kirchhoff's current law

$$I_1 + I_5 = I_2$$

$$\therefore (V_b - V_a)/R_2 + V_b/R_4 + (V_b - V_2)/R_5 = 0 \quad (3.40)$$

Rearranging the above equations we get

$$(1/R_1 + 1/R_2 + 1/R_3)V_a - (1/R_3)V_b = (1/R_1)V_1 \quad (3.41)$$

$$(-1/R_2)V_a + (1/R_2 + 1/R_4 + 1/R_5)V_b = V_2/R_5 \quad (3.42)$$

In general, the above equation can be written as

$$G_{aa}V_a + G_{ab}V_b = I_1 \quad (3.43)$$

$$G_{ba}V_a + G_{bb}V_b = I_2 \quad (3.44)$$

By comparing Eqs 3.41, 3.42 and Eqs 3.43, 3.44 we have the self conductance at node a, $G_{aa} = (1/R_1 + 1/R_2 + 1/R_3)$ is the sum of the conductances connected to node a. Similarly, $G_{bb} = (1/R_2 + 1/R_4 + 1/R_5)$ is the sum of the conductances connected to node b. $G_{ab} = (-1/R_3)$ is the sum of the mutual conductances connected to node a and node b. Here all the mutual conductances have negative signs. Similarly, $G_{ba} = (-1/R_3)$ is also a mutual conductance connected between nodes b and a. I_1 and I_2 are the sum of the source currents at node a and node b, respectively. The current which drives into the node has positive sign, while the current that drives away from the node has negative sign.

Example 3.8 for the circuit shown in the figure 3.18 write the node equations by the inspection method.

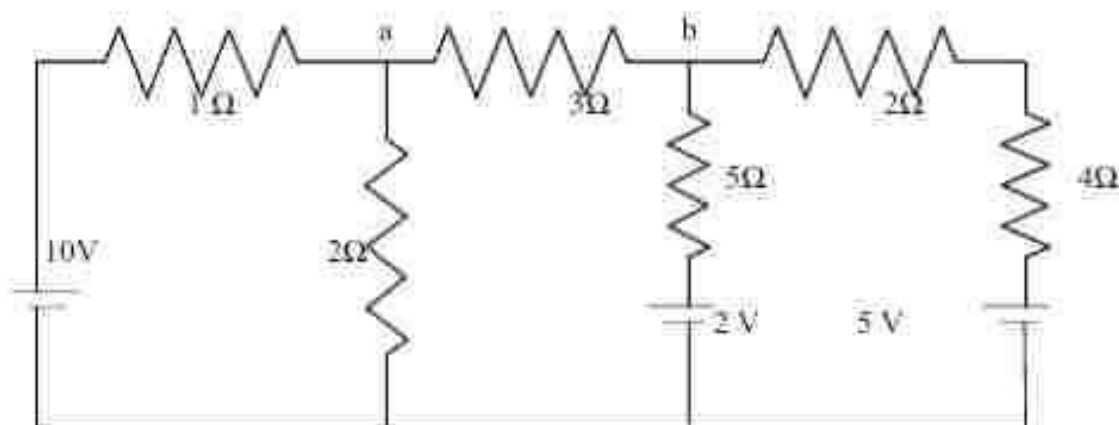


Fig 3.18

Solution:-

The general equations are

$$G_{aa}V_a + G_{ab}V_b = I_1 \quad (3.45)$$

$$G_{ba}V_a + G_{bb}V_b = I_2 \quad (3.46)$$

Consider equation 3.45

$G_{aa} = (1 + 1/2 + 1/3)$ mho. The self conductance at node a is the sum of the conductances connected to node a .

$G_{bb} = (1/6 + 1/5 + 1/3)$ mho the self conductance at node b is the sum of conductances connected to node b .

$G_{ab} = -(1/3)$ mho, the mutual conductances between nodes a and b is the sum of the conductances connected between node a and b .

Similarly, $G_{ba} = -(1/3)$, the sum of the mutual conductances between nodes b and a .

$I_1 = 10/1 = 10$ A, the source current at node a .

$I_2 = (2/5 + 5/6) = 1.23\text{A}$, the source current at node b .

Therefore, the nodal equations are

$$1.83V_a - 0.33V_b = 10 \quad (3.47)$$

$$-0.33V_a + 0.7V_b = 1.23 \quad (3.48)$$

3.7 SUPERNODE ANALYSIS

Suppose any of the branches in the network has a voltage source, then it is slightly difficult to apply nodal analysis. One way to overcome this difficulty is to apply the supernode technique. In this method, the two adjacent nodes that are connected by a voltage source are reduced to a single node and then the equations are formed by applying Kirchhoff's current law as usual. This is explained with the help of fig. 3.19

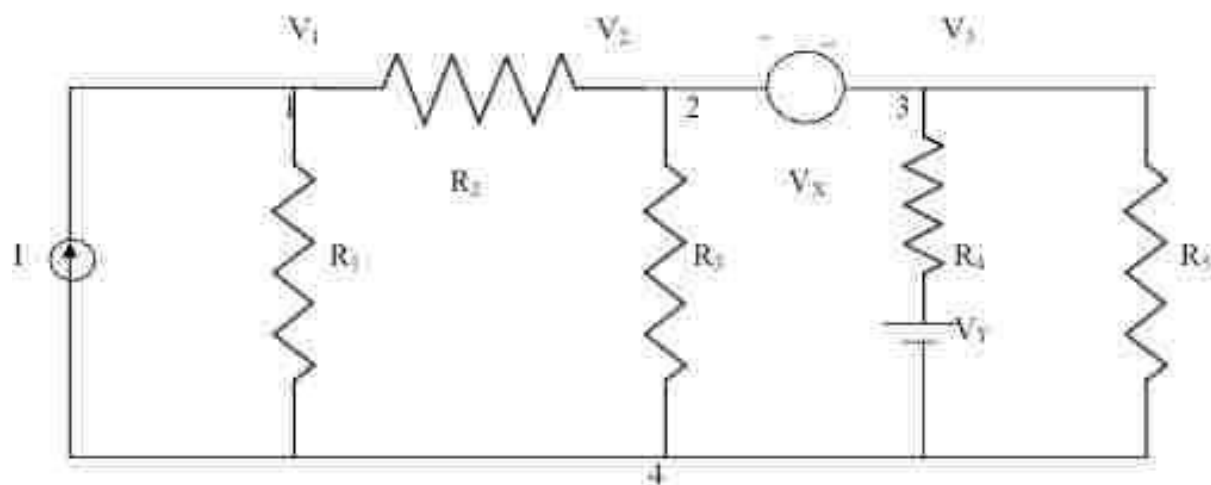


FIG 3.19

It is clear from the fig.3.19, that node 4 is the reference node. Applying Kirchhoff's current law at node 1, we get

$$I = (V_1/R_1) + (V_1 - V_2)/R_2$$

Due to the presence of voltage source V_x in between nodes 2 and 3, it is slightly difficult to find out the current. The supernode technique can be conveniently applied in this case.

Accordingly, we can write the combined equation for nodes 2 and 3 as under.

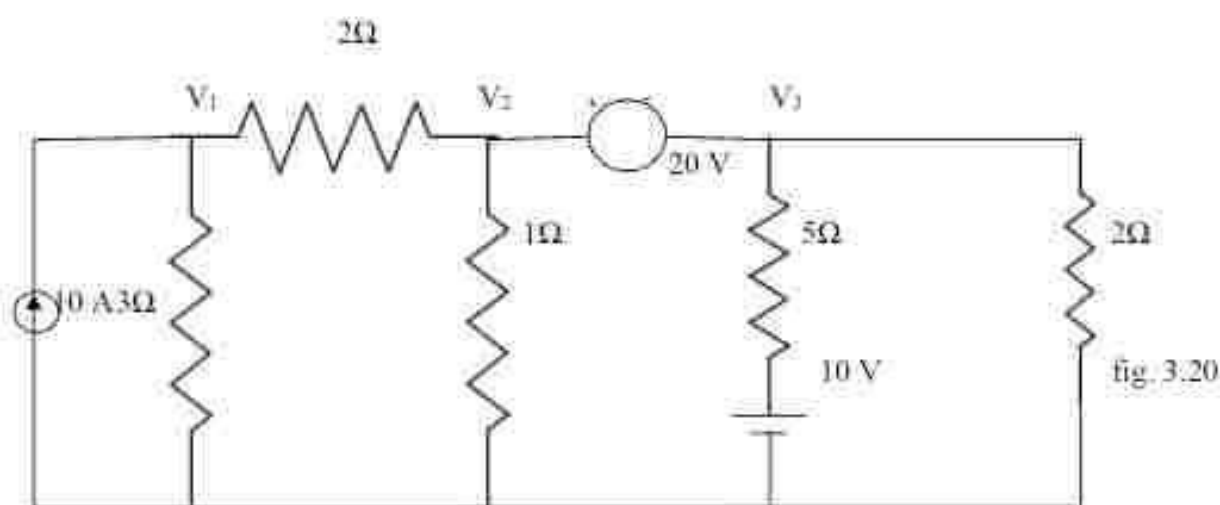
$$(V_2 - V_1)/R_2 + V_2/R_3 + (V_3 - V_2)/R_4 + V_3/R_5 = 0$$

The other equation is

$$V_2 - V_3 = V_s$$

From the above three equations, we can find the three unknown voltages.

Example 3.9 Determine the current in the $5\ \Omega$ resistor for the circuit shown in fig. 3.20



Solution. At node 1

$$10 = V_1/3 + (V_1 - V_2)/2$$

$$\text{Or } V_1[1/3 + 1/2] - (V_2/2) - 10 = 0$$

$$0.83V_1 - 0.5V_2 - 10 = 0 \quad (3.49)$$

At node 2 and 3, the supernode equation is

$$(V_2 - V_1)/2 + V_2/1 + (V_2 - 10)/5 + V_3/2 = 0$$

$$\text{Or } -V_1/2 + V_2[(1/2) + 1] + V_3[1/5 + 1/2] = 2$$

$$\text{Or } -0.5V_1 + 1.5V_2 + 0.7V_3 - 2 = 0 \quad (2.50)$$

The voltage between nodes 2 and 3 is given by

$$V_2 - V_3 = 20 \quad (3.51)$$

Chapter-04

NETWORK THEORM

INTRODUCTION

This chapter introduces a number of theorems that have application throughout the field of electricity and electronics. Not only can they be used to solve networks such as encountered in the previous chapter, but they also provide an opportunity to determine the impact of a particular source or element on the response of the entire system. In most cases, the network to be analyzed and the mathematics required to find the solution are simplified. All of the theorems appear again in the analysis of ac networks. In fact, the application of each theorem to ac networks is very similar in content to that found in this chapter.

The first theorem to be introduced is the superposition theorem, followed by Thévenin's theorem, Norton's theorem, and the maximum power transfer theorem. The chapter concludes with a brief introduction to Millman's theorem and the substitution and reciprocity theorems.

SUPERPOSITION THEOREM

The superposition theorem states that "The current through, or voltage across, any element of a network is equal to the algebraic sum of the currents or voltages produced independently by each source."

In other words, this theorem allows us to find a solution for a current or voltage using only one source at a time. Once we have the solution for each source, we can combine the results to obtain the total solution. The term algebraic appears in the above theorem statement because the currents resulting from the sources of the network can have different directions, just as the resulting voltages can have opposite polarities.

If we are to consider the effects of each source, the other sources obviously must be removed. Setting a voltage source to zero volts is like placing a short circuit across its terminals. Therefore, when removing a voltage source from a network schematic, replace it with a direct connection (short circuit) of zero ohms. Any internal resistance associated with the source must remain in the network.

Setting a current source to zero amperes is like replacing it with an open circuit. Therefore, when removing a current source from a network schematic, replace it by an open circuit of infinite ohms. Any internal resistance associated with the source must remain in the network.

The above statements are illustrated in Fig.

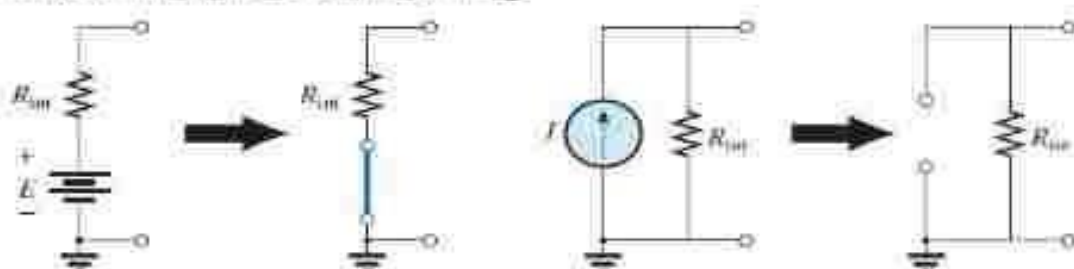


FIG. 9.1

Removing a voltage source and a current source to permit the application of the superposition theorem.

EXAMPLE 9.1

a. Using the superposition theorem, determine the current through resistor R_2 for the network in Fig. 9.2.

Solution:

In order to determine the effect of the 36 V voltage source, the current source must be replaced by an open-circuit equivalent as shown in Fig. 9.3. The result is a simple series circuit with a current equal to

$$I_2' = \frac{E}{R_T} = \frac{E}{R_1 + R_2} = \frac{36 \text{ V}}{12 \Omega + 6 \Omega} = \frac{36 \text{ V}}{18 \Omega} = 2 \text{ A}$$

Examining the effect of the 9 A current source requires replacing the 36 V voltage source by a short-circuit equivalent as shown in Fig. 9.4. The result is a parallel combination of resistors R_1 and 2. Applying the current divider rule results in

$$I_2'' = \frac{R_1(I)}{R_1 + R_2} = \frac{(12 \Omega)(9 \text{ A})}{12 \Omega + 6 \Omega} = 6 \text{ A}$$

Since the contribution to current I_2 has the same direction for each source, as shown in Fig. 9.5, the total solution for current I_2 is the sum of the currents established by the two sources. That is,

$$I_2 = I_2' + I_2'' = 2 \text{ A} + 6 \text{ A} = 8 \text{ A}$$

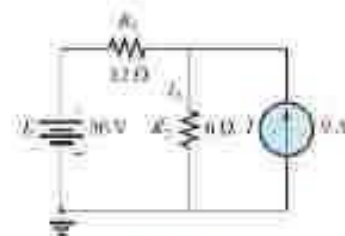


FIG. 9.2

Network to be analyzed in Example 9.1 using the superposition theorem.

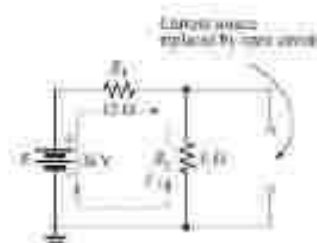


FIG. 9.3

Replacing the 9 A current source in Fig. 9.2 by an open circuit to determine the effect of the 36 V voltage source on current I_2 .

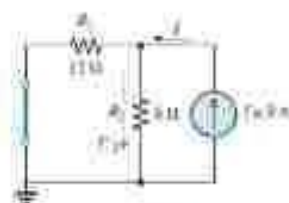


FIG. 9.4

Replacing the 36 V voltage source by a short-circuit equivalent to determine the effect of the 9 A current source on current I_2 .

EXAMPLE 9.2 Using the superposition theorem, determine the current through the 12 Ω resistors in Fig. 9.8. Note that this is a two-source network of the type examined in the previous chapter when we applied branch-current analysis and mesh analysis.

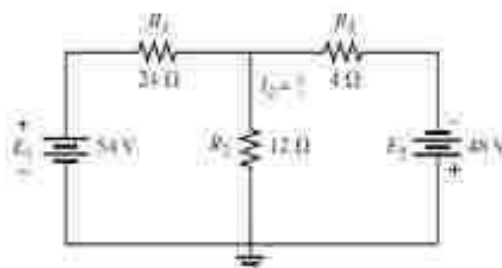


FIG. 9.8

Using the superposition theorem to determine the current through the 12-Ω resistor (Example 9.2).

Solution: Considering the effects of the 54 V source requires replacing the 48 V source by a short-circuit equivalent as shown in Fig. 9.9. The result is that the 12 Ω and 4 Ω resistors are in parallel. The total resistance seen by the source is therefore,

$$R_T = R_1 + R_2 \parallel R_3 = 24 \Omega + 12 \Omega \parallel 4 \Omega = 24 \Omega + 3 \Omega = 27 \Omega$$

and the source current is

$$I_1 = \frac{E_1}{R_T} = \frac{54 \text{ V}}{27 \Omega} = 2 \text{ A}$$

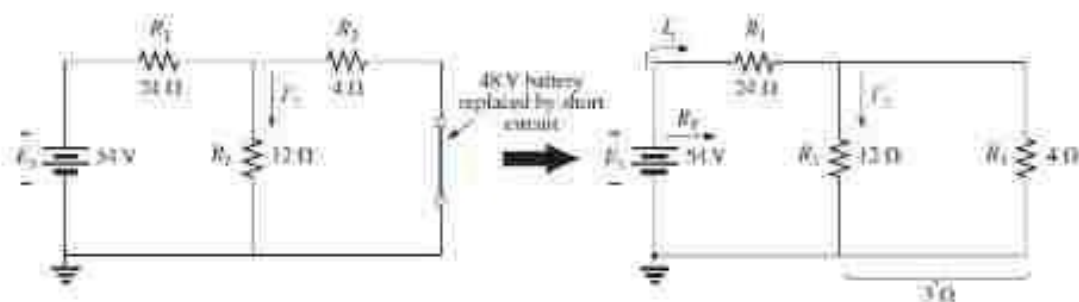


FIG. 9.9

Using the superposition theorem to determine the effect of the 54 V voltage source on current I_2 in Fig. 9.8.

Using the current divider rule results in the contribution to I_2 due to the 54 V source:

$$I_2' = \frac{R_3 I_s}{R_2 + R_3} = \frac{(4 \Omega)(2 \text{ A})}{4 \Omega + 12 \Omega} = 0.5 \text{ A}$$

If we now replace the 54 V source by a short-circuit equivalent, the network in Fig. 9.10 results. The result is a parallel connection for the 12 Ω and 24 Ω resistors.

Therefore, the total resistance seen by the 48 V source is

$$R_T = R_3 + R_2 \parallel R_1 = 4 \Omega + 12 \Omega \parallel 24 \Omega = 4 \Omega + 8 \Omega = 12 \Omega$$

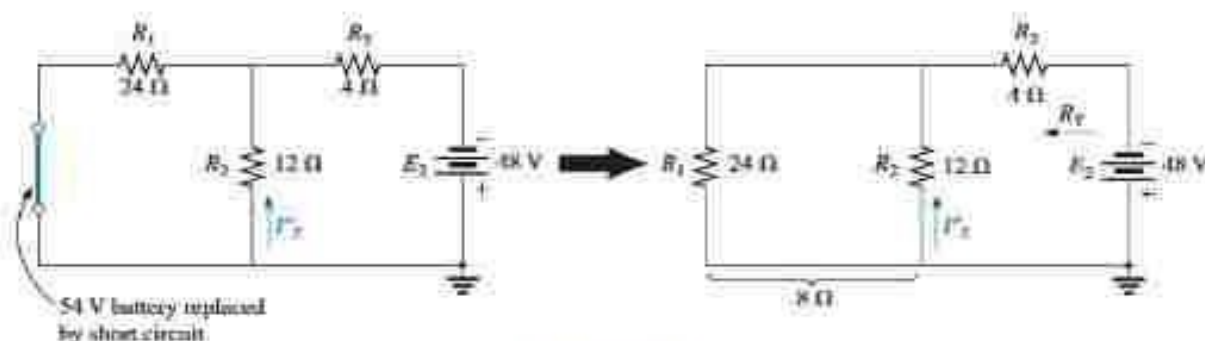


FIG. 9.10

Using the superposition theorem to determine the effect of the 48 V voltage source on current I_2 in Fig. 9.8.

and the source current is

$$I_s = \frac{E_2}{R_T} = \frac{48 \text{ V}}{12 \Omega} = 4 \text{ A}$$

Applying the current divider rule results in

$$I_2'' = \frac{R_1(I_s)}{R_1 + R_2} = \frac{(24 \Omega)(4 \text{ A})}{24 \Omega + 12 \Omega} = 2.67 \text{ A}$$

It is now important to realize that current I_2 due to each source has a different direction, as shown in Fig. 9.11. The net current therefore is the difference of the two and in the direction of the larger as follows:

$$I_2 = I_2'' - I_2' = 2.67 \text{ A} - 0.5 \text{ A} = 2.17 \text{ A}$$

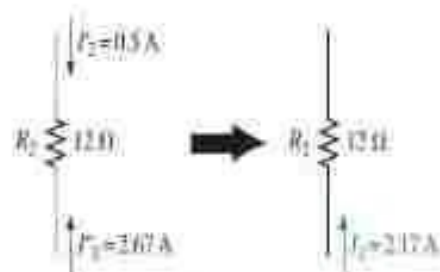


FIG. 9.11

Using the results of Figs. 9.9 and 9.10 to determine current I_2 for the network in Fig. 9.8.

EXAMPLE 9.3 Using the superposition theorem, determine current I_1 for the network in

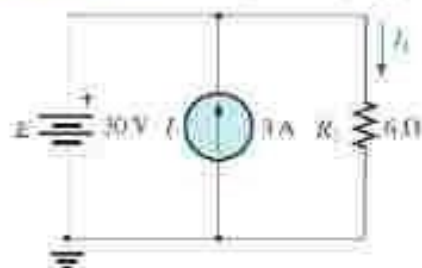


FIG. 9.12

Two-source network to be analyzed using the superposition theorem in Example 9.3.

Solution: Since two sources are present, there are two networks to be analyzed. First let us determine the effects of the voltage source by setting the current source to zero amperes as shown in Fig. 9.13. Note that the resulting current is defined as I_1' because it is the current through resistor R_1 due to the voltage source only.

Due to the open circuit, resistor R_1 is in series (and, in fact, in parallel) with the voltage source E . The voltage across the resistor is the applied voltage, and current I_1' is determined by

$$I_1' = \frac{V_1}{R_1} = \frac{E}{R_1} = \frac{30 \text{ V}}{6 \Omega} = 5 \text{ A}$$

Now for the contribution due to the current source. Setting the voltage source to zero volts results in the network in Fig. 9.14; this presents us with an interesting situation. The current source has been replaced with a short-circuit equivalent that is directly across the current source and resistor R_1 . Since the source current takes the path of least resistance, it

chooses the zero ohm path of the inserted short-circuit equivalent, and the current through R_1 is zero amperes. This is clearly demonstrated by an application of the current divider rule as follows:

$$I_1'' = \frac{R_{sc} I}{R_{sc} + R_1} = \frac{(0 \Omega) I}{0 \Omega + 6 \Omega} = 0 \text{ A}$$

Since I_1' and I_1'' have the same defined direction in Figs. 9.13 and 9.14, the total current is defined by

$$I_1 = I_1' + I_1'' = 5 \text{ A} + 0 \text{ A} = 5 \text{ A}$$

Although this has been an excellent introduction to the application of the superposition theorem, it should be immediately clear in Fig. 9.12 that the voltage source is in parallel with the current source and load resistor R_1 , so the voltage across each must be 30 V. The result is that I_1 must be determined solely by

$$I_1 = \frac{V_1}{R_1} = \frac{E}{R_1} = \frac{30 \text{ V}}{6 \Omega} = 5 \text{ A}$$



FIG. 9.13

Determining the effect of the 30 V supply on the current I_1 in Fig. 9.12.

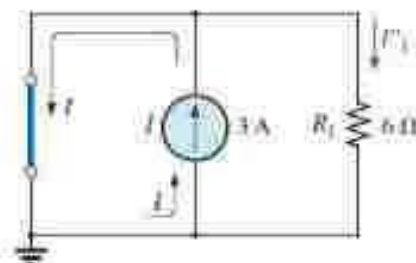


FIG. 9.14

Determining the effect of the 3 A current source on the current I_1 in Fig. 9.12.

EXAMPLE 9.4 Using the principle of superposition, find the current I_2 through the $12\text{ k}\Omega$ resistor in Fig. 9.15.

Solution: Consider the effect of the 6 mA current source (Fig. 9.16).

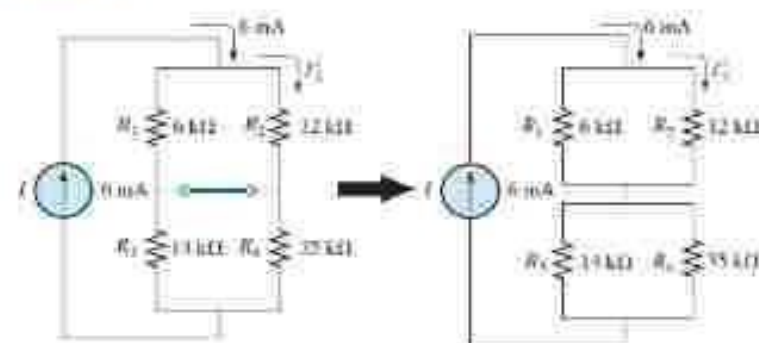


FIG. 9.16

The effect of the current source I on the current I_2 .

The current divider rule gives

$$I_2' = \frac{R_1 I}{R_1 + R_2} = \frac{(6\text{ k}\Omega)(6\text{ mA})}{6\text{ k}\Omega + 12\text{ k}\Omega} = 2\text{ mA}$$

Considering the effect of the 9 V voltage source (Fig. 9.17) gives

$$I_2'' = \frac{E}{R_1 + R_2} = \frac{9\text{ V}}{6\text{ k}\Omega + 12\text{ k}\Omega} = 0.5\text{ mA}$$

Since I_2' and I_2'' have the same direction through R_2 , the desired current is the sum of the two:

$$\begin{aligned} I_2 &= I_2' + I_2'' \\ &= 2\text{ mA} + 0.5\text{ mA} \\ &= 2.5\text{ mA} \end{aligned}$$

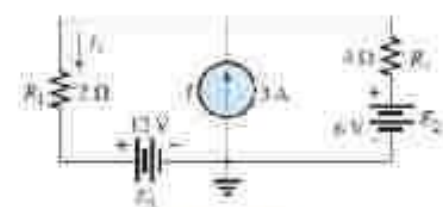
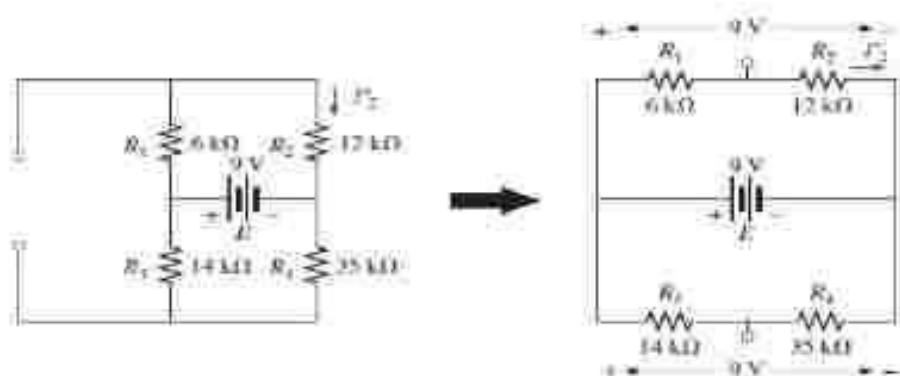


FIG. 9.18
Example 9.5.

EXAMPLE 9.5: Find the current through the $2\text{ }\Omega$ resistor of the network in Fig. 9.18. The presence of three sources results in three different networks to be analyzed.

Solution: Consider the effect of the 12 V source (Fig. 9.19).

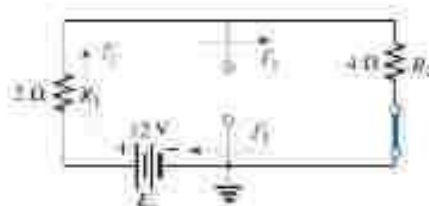


FIG. 9.19

The effect of E_1 on the current I_1 .



FIG. 9.20

The effect of E_2 on the current I_1 .

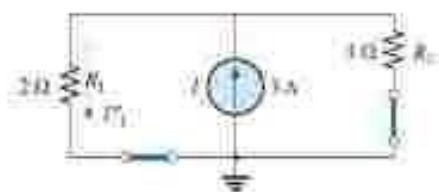


FIG. 9.21

The effect of I_1 on the current I_1 .

$$I_1' = \frac{E_1}{R_1 + R_2} = \frac{12\text{ V}}{2\ \Omega + 4\ \Omega} = \frac{12\text{ V}}{6\ \Omega} = 2\text{ A}$$

Consider the effect of the 6 V source (Fig. 9.20):

$$I_1'' = \frac{E_2}{R_1 + R_2} = \frac{6\text{ V}}{2\ \Omega + 4\ \Omega} = \frac{6\text{ V}}{6\ \Omega} = 1\text{ A}$$

Consider the effect of the 3 A source (Fig. 9.21). Applying the current divider rule gives:

$$I_1''' = \frac{R_2 I}{R_1 + R_2} = \frac{(4\ \Omega)(3\text{ A})}{2\ \Omega + 4\ \Omega} = \frac{12\text{ A}}{6} = 2\text{ A}$$

The total current through the 2 ohm resistor appears in Fig. 9.22, and

$$I_1 = \overbrace{I_1' + I_1''}^{\text{Same direction as } I_1 \text{ in Fig. 9.18}} + \overbrace{I_1'''}^{\text{Opposite direction to } I_1 \text{ in Fig. 9.18}} = 1\text{ A} + 2\text{ A} - 2\text{ A} = 1\text{ A}$$

9.3 THÉVENIN'S THEOREM

The next theorem to be introduced, **Thévenin's theorem**, is probably one of the most interesting in that it permits the reduction of complex networks to a simpler form for analysis and design.

In general, the theorem can be used to do the following:

- Analyze networks with sources that are not in series or parallel.
- Reduce the number of components required to establish the same characteristics at the output terminals.
- Investigate the effect of changing a particular component on the behaviour of a network without having to analyze the entire network after each change.

All three areas of application are demonstrated in the examples to follow.

Thévenin's theorem states the following:

Any two-terminal dc network can be replaced by an equivalent circuit consisting solely of a voltage source and a series resistor as shown in

Fig. 9.23.

The theorem was developed by Commandant Leon-Charles Thévenin in 1883 as described in Fig. 9.24.

To demonstrate the power of the theorem, consider the fairly complex network of Fig. 9.25(a) with its two sources and series-parallel connections.

The theorem states that the entire network inside the blue shaded area can be replaced by one voltage source and one resistor as shown in Fig. 9.25(b). If the replacement is done properly, the voltage across, and the current through, the resistor R_L will be the same for each network. The value of R_L can be changed to any

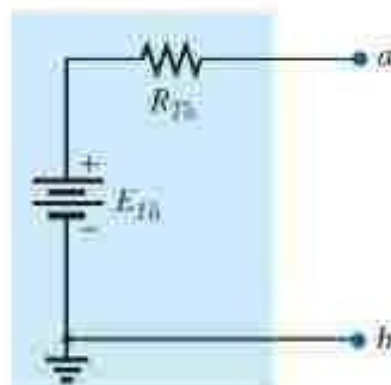


FIG. 9.23

Thévenin equivalent circuit.



FIG. 9.24

*Leon-Charles Thévenin,
Commandant of the Polytechnique Ecole
Polytechnique, Paris, France.*

value, and the voltage, current, or power to the load resistor is the same for each configuration.

Now, this is a very powerful statement—one that is verified in the examples to follow.

The question then is, How can you determine the proper value of Thévenin voltage and resistance? In general, finding the Thévenin *resistance* value is quite straightforward. Finding the Thévenin *voltage* can be more of a challenge and, in fact, may require using the superposition theorem.

Fortunately, there is a series of steps that will lead to the proper value of each parameter. Although a few of the steps may seem trivial at first, they can become quite important when the network becomes complex.

Thévenin's Theorem Procedure

Preliminary:

1. Remove that portion of the network where the Thévenin equivalent circuit is found. In Fig. 9.25(a), this requires that the load resistor R_L be temporarily removed from the network.

2. Mark the terminals of the remaining two-terminal network. (The importance of this step will become obvious as we progress through some complex networks.) R_{Th} :

3. Calculate R_{Th} by first setting all sources to zero (voltage sources are replaced by short circuits and current sources by open circuits) and then finding the resultant resistance between the two marked terminals. (If the internal resistance of the voltage and/or current sources is included in the original network, it must remain when the sources are set to zero.) E_{Th} :

4. Calculate E_{Th} by first returning all sources to their original position and finding the open-circuit voltage between the marked terminals. (This step is invariably the one that causes most confusion and errors. In all cases, keep in mind that it is the open-circuit potential between the two terminals marked in step 2.) *Conclusion:*

5. Draw the Thévenin equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the equivalent circuit. This step is indicated by the placement of the resistor R_L between the terminals of the Thévenin equivalent circuit as shown in Fig. 9.25(b).

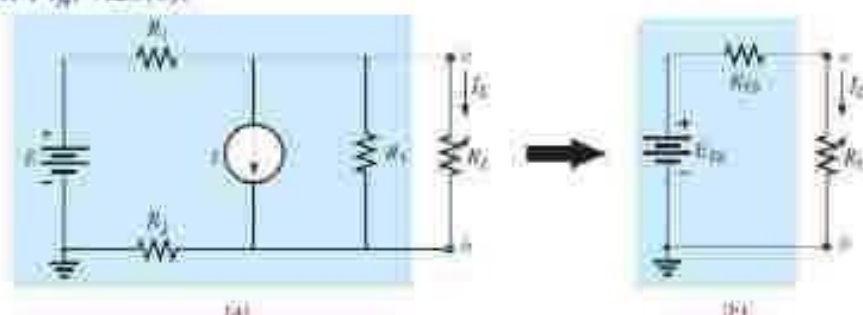


FIG. 9.25

Substituting the Thévenin equivalent circuit for a complex network

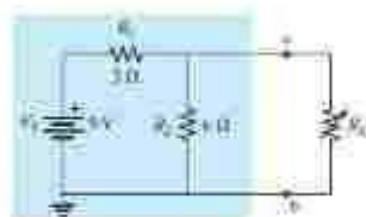


FIG. 9.26
Example 9.6.

EXAMPLE 9.6 Find the Thévenin equivalent circuit for the network in the shaded area of the network in Fig. 9.26. Then find the current through R_L for values of $2\ \Omega$, $10\ \Omega$, and $100\ \Omega$.

Solution:

Steps 1 and 2: These produce the network in Fig. 9.27. Note that the load resistor R_L has been removed and the two "holding" terminals have been defined as a and b .

Step 3: Replacing the voltage source E_1 with a short-circuit equivalent yields the network in Fig. 9.28(a), where

$$R_{Th} = R_1 \parallel R_2 = \frac{(3 \Omega)(6 \Omega)}{3 \Omega + 6 \Omega} = 2 \Omega$$

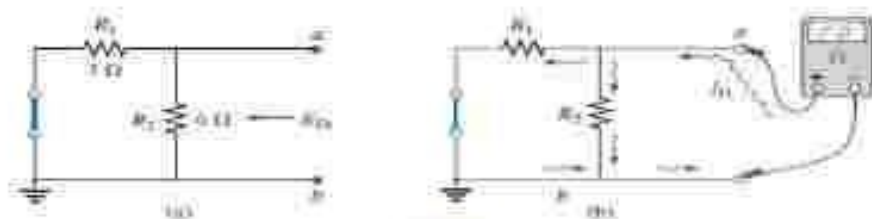


FIG. 9.28
Determining R_{Th} for the network in Fig. 9.27.

The importance of the two marked terminals now begins to surface. They are the two terminals across which the Thévenin resistance is measured. It is no longer the total resistance as seen by the source, as determined in the majority of problems of Chapter 7. If some difficulty develops when determining R_{Th} with regard to whether the resistive elements are in series or parallel, consider recalling that the ohmmeter sends out a trickle current into a resistive combination and senses the

level of the resulting voltage to establish the measured resistance level. In Fig. 9.28(b), the trickle current of the ohmmeter approaches the network through terminal a , and when it reaches the junction of R_1 and R_2 , it splits as shown. The fact that the trickle current splits and then recombines at the lower node reveals that the resistors are in parallel as far as the ohmmeter reading is concerned. In essence, the path of the sensing current of the ohmmeter has revealed how the resistors are connected to the two terminals of interest and how the Thévenin resistance should be determined. Remember this as you work through the various examples in this section.

Step 4: Replace the voltage source (Fig. 9.29). For this case, the open-circuit voltage E_{Th} is the same as the voltage drop across the 6 Ω resistor.

Applying the voltage divider rule gives

$$E_{Th} = \frac{R_2 E_1}{R_2 + R_1} = \frac{(6 \Omega)(9 \text{ V})}{6 \Omega + 3 \Omega} = \frac{54 \text{ V}}{9} = 6 \text{ V}$$

It is particularly important to recognize that E_{Th} is the open-circuit potential between points a and b . Remember that an open circuit can have any voltage across it, but the current must be zero. In fact, the current through any element in series with the open circuit must be zero also. The use of a voltmeter to measure E_{Th} appears in Fig. 9.30. Note that it is placed directly across the resistor R_2 since E_{Th} and V_{R_2} are in parallel.

Step 5: (Fig. 9.31):

$$\begin{aligned} I_L &= \frac{E_{Th}}{R_{Th} + R_L} \\ R_L = 2 \Omega: \quad I_L &= \frac{6 \text{ V}}{2 \Omega + 2 \Omega} = 1.5 \text{ A} \\ R_L = 10 \Omega: \quad I_L &= \frac{6 \text{ V}}{2 \Omega + 10 \Omega} = 0.5 \text{ A} \\ R_L = 100 \Omega: \quad I_L &= \frac{6 \text{ V}}{2 \Omega + 100 \Omega} = 0.06 \text{ A} \end{aligned}$$

If Thévenin's theorem were unavailable, each change in R_L would require that the entire network in Fig. 9.26 be reexamined to find the new value of I_L .

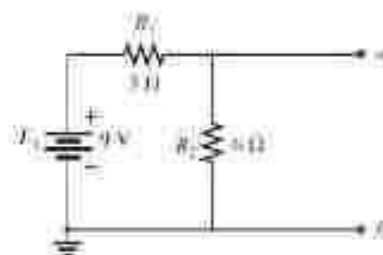


FIG. 9.27

Identifying the terminals of particular importance when applying Thévenin's theorem.

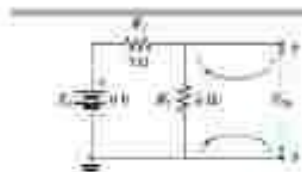


FIG. 9.28
Determining R_{Th} for the network in Fig. 9.27.

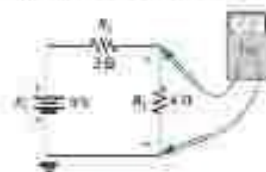


FIG. 9.29
Measuring E_{Th} for the network in Fig. 9.27.



FIG. 9.31
Substituting the Thévenin equivalent circuit for the network external to R_L in Fig. 9.26.

EXAMPLE 9.7 Find the Thévenin equivalent circuit for the network in the shaded area of the network in Fig. 9.32.

Solution:

Steps 1 and 2: See Fig. 9.33.

Step 3: See Fig. 9.34. The current source has been replaced with an open-circuit equivalent and the resistance determined between terminals *a* and *b*.

In this case, an ohmmeter connected between terminals *a* and *b* sends out a sensing current that flows directly through R_1 and R_2 (at the same level). The result is that R_1 and R_2 are in series and the Thévenin resistance is the sum of the two.

$$R_{Th} = R_1 + R_2 = 4\ \Omega + 2\ \Omega = 6\ \Omega$$

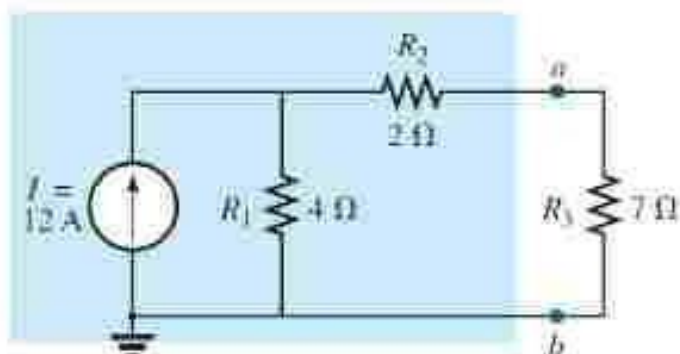


FIG. 9.32

Example 9.7.

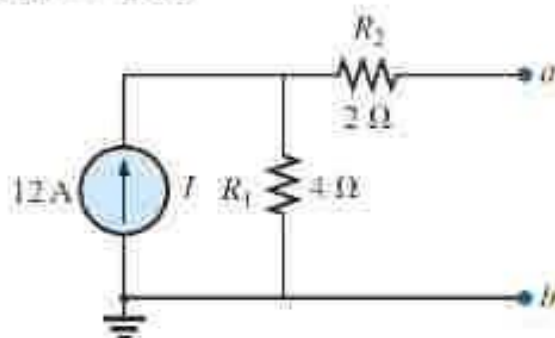


FIG. 9.33

Establishing the terminals of particular interest for the network in Fig. 9.32.

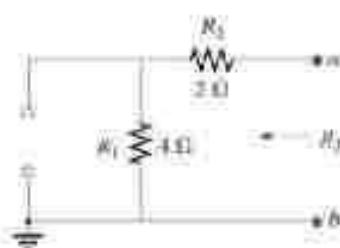


FIG. 9.34

Determining R_{Th} for the network in Fig. 9.32.

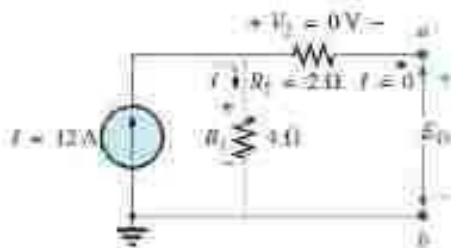


FIG. 9.35

Determining E_{Th} for the network in Fig. 9.32.

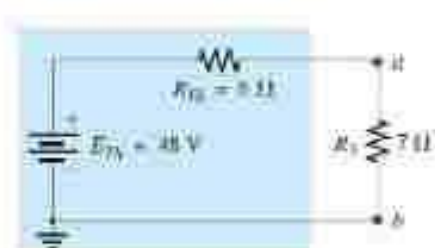


FIG. 9.36

Substituting the Thévenin equivalent circuit in the network external to the resistor R_L in Fig. 9.32.

Step 4: See Fig. 9.35. In this case, since an open circuit exists between the two marked terminals, the current is zero between these terminals and through the $2\ \Omega$ resistor. The voltage drop across R_2 is, therefore,

$$V_2 = I_2 R_2 = (0) R_2 = 0\ \text{V}$$

and

$$E_{Th} = V_1 = I_1 R_1 = I R_1 = (12\ \text{A})(4\ \Omega) = 48\ \text{V}$$

Step 5: See Fig. 9.36.

EXAMPLE 9.8 Find the Thévenin equivalent circuit for the network in the shaded area of the network in Fig. 9.37. Note in this example that there is no need for the section of the network to be preserved to be at the "end" of the configuration.

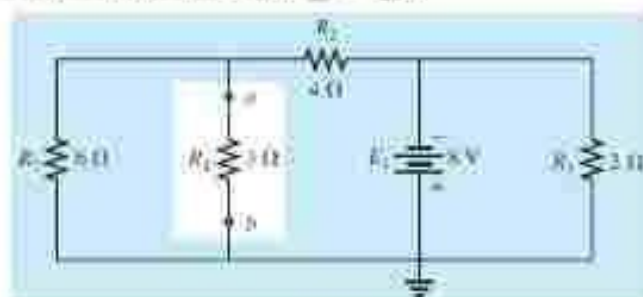


FIG. 9.37

Solution:

Steps 1 and 2: See Fig. 9.38

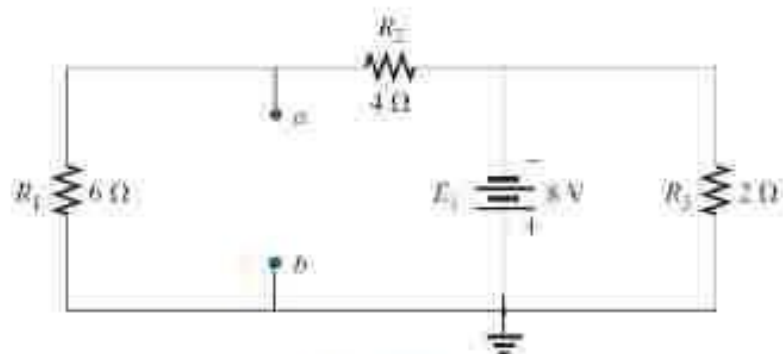


FIG. 9.38

Identifying the terminals of particular interest for the network in Fig. 9.37.

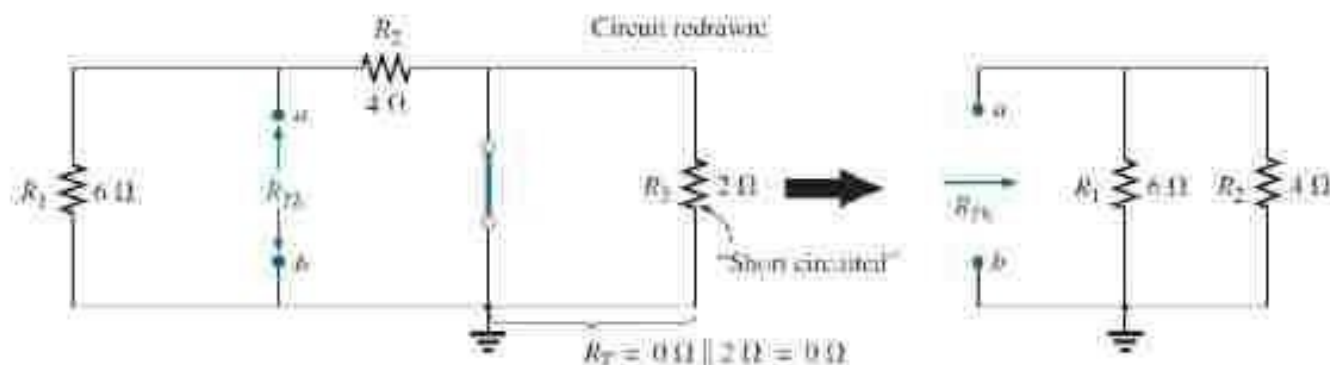


FIG. 9.39

Determining R_{Th} for the network in Fig. 9.38.

Step 3: See Fig. 9.39. Steps 1 and 2 are relatively easy to apply, but now we must be careful to "hold" onto the terminals a and b as the Thévenin resistance and voltage are determined. In Fig. 9.39, all the remaining elements turn out to be in parallel, and the network can be redrawn as shown. We have

$$R_{Th} = R_1 \parallel R_2 = \frac{(6 \Omega)(4 \Omega)}{6 \Omega + 4 \Omega} = \frac{24 \Omega}{10} = 2.4 \Omega$$

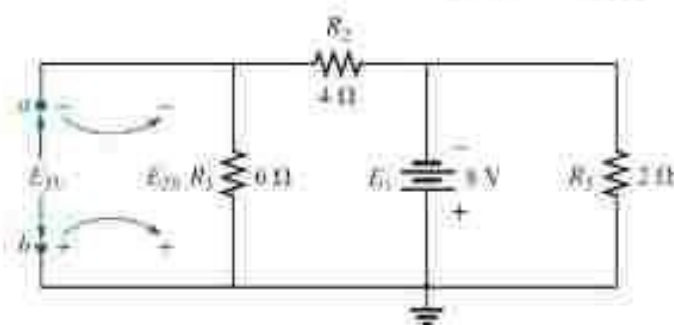


FIG. 9.40

Determining E_{Th} for the network in Fig. 9.38.

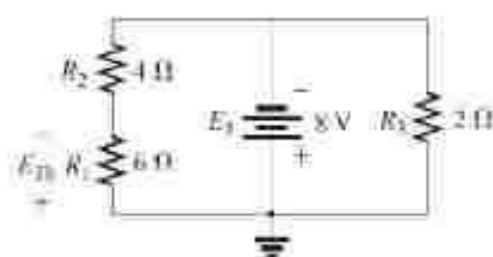


FIG. 9.41

Network of Fig. 9.40 redrawn.

Step 4: See Fig. 9.40. In this case, the network can be redrawn as shown in Fig. 9.41. Since the voltage is the same across parallel elements, the voltage across the series resistors R_1 and R_2 is E_1 , or 8 V. Applying the voltage divider rule gives

$$E_{Th} = \frac{R_1 E_1}{R_1 + R_2} = \frac{(6 \Omega)(8 \text{ V})}{6 \Omega + 4 \Omega} = \frac{48 \text{ V}}{10} = 4.8 \text{ V}$$

Step 3: See Fig. 9.42.

EXAMPLE 9.9 Find the Thévenin equivalent circuit for the network in the shaded area of the bridge network in Fig. 9.43.

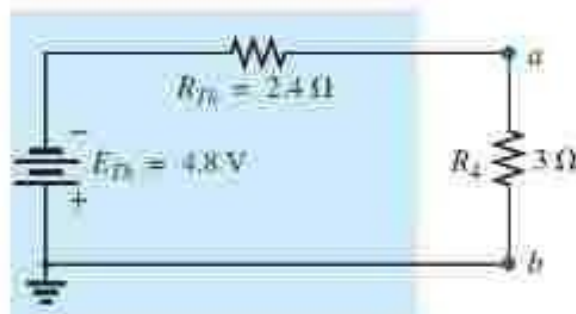


FIG. 9.42

Substituting the Thévenin equivalent circuit for the network external to the resistor R_L in Fig. 9.37.

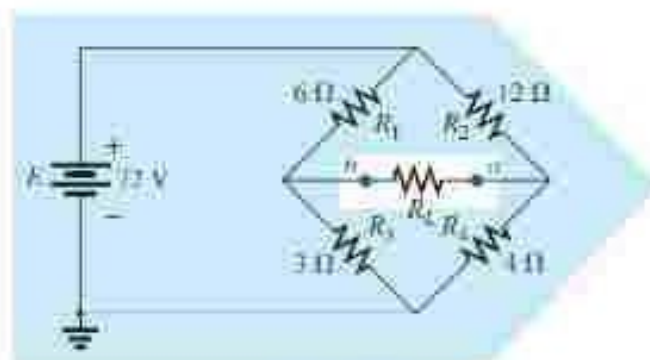


FIG. 9.43

Example 9.9.

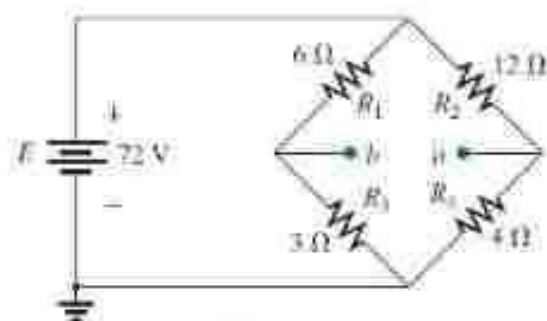


FIG. 9.44

Identifying the terminals of particular interest for the network in Fig. 9.43.

Solution:

Steps 1 and 2: See Fig. 9.44.

Step 3: See Fig. 9.45. In this case, the short-circuit replacement of the voltage source E provides a direct connection between c and c' in Fig. 9.45(a), permitting a "folding" of the network around the horizontal line of $a-b$ to produce the configuration in Fig. 9.45(b).

$$\begin{aligned} R_{Th} &= R_{a-b} = R_1 \parallel R_3 + R_2 \parallel R_4 \\ &= 6 \Omega \parallel 3 \Omega + 4 \Omega \parallel 12 \Omega \\ &= 2 \Omega + 3 \Omega = 5 \Omega \end{aligned}$$

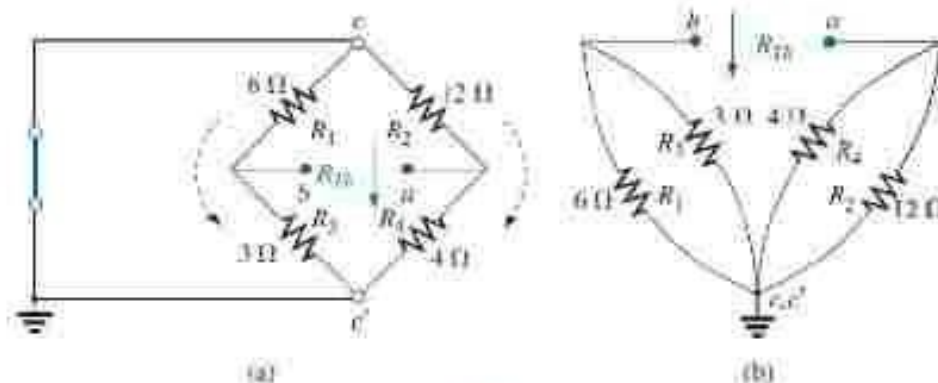


FIG. 9.45

Solving for R_{Th} for the network in Fig. 9.44.

Step 4: The circuit is redrawn in Fig. 9.46. The absence of a direct connection between a and b results in a network with three parallel branches. The voltages V_1 and V_2 can therefore be determined using the voltage divider rule:

$$V_1 = \frac{R_1 E}{R_1 + R_3} = \frac{(6 \Omega)(72 \text{ V})}{6 \Omega + 3 \Omega} = \frac{432 \text{ V}}{9} = 48 \text{ V}$$

$$V_2 = \frac{R_2 E}{R_2 + R_4} = \frac{(12 \Omega)(72 \text{ V})}{12 \Omega + 4 \Omega} = \frac{864 \text{ V}}{16} = 54 \text{ V}$$

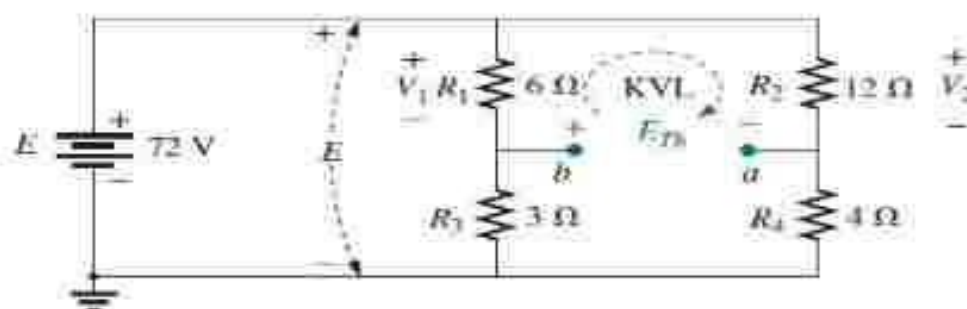


FIG. 9.46

Determining E_{Th} for the network in Fig. 9.44.

Assuming the polarity shown for E_{Th} and applying Kirchhoff's voltage law to the top loop in the clockwise direction results in

$$\sum_v V = +E_{Th} + V_1 - V_2 = 0$$

and
$$E_{Th} = V_2 - V_1 = 54 \text{ V} - 48 \text{ V} = 6 \text{ V}$$

Step 5: See Fig. 9.47.

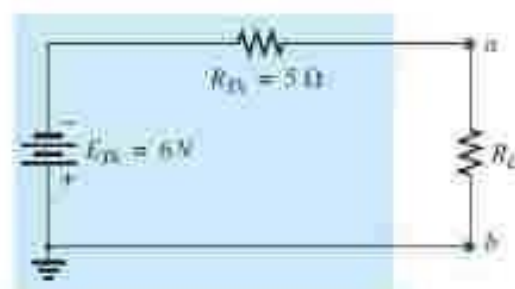


FIG. 9.47

EXAMPLE 9.11 For the network of Fig. 9.54, a. Find the Thévenin equivalent circuit for the portion of the network in the shaded area. b. Reconstruct the network of Fig. 9.54 with the Thévenin equivalent network in place. c. Using the resulting network of part (b) find the voltage V_a .

Solutions:

a. *Steps 1 and 2:* See Fig. 9.55.

Step 3: See Fig. 9.56.

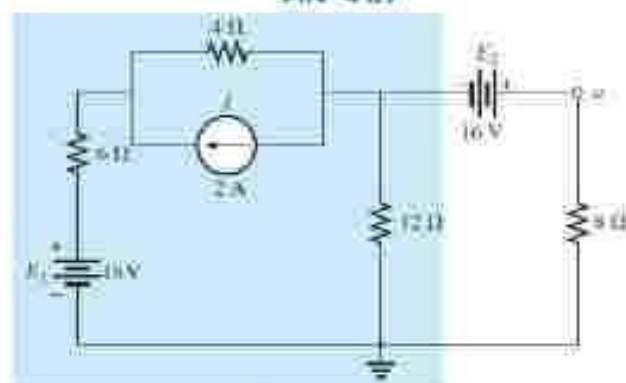


FIG. 9.54
Example 9.11.

$$R_{Th} = 12 \Omega \parallel (4 \Omega + 6 \Omega) = 12 \Omega \parallel 10 \Omega = 5$$

Step 4: Applying the superposition theorem, we will first find the effect of the voltage source on the Thévenin voltage using the network of Fig. 9.57. Applying the voltage divider rule:

$$E'_{Th} = \frac{12 \Omega (18 \text{ V})}{6 \Omega + 4 \Omega + 12 \Omega} = \frac{216}{22} \text{ V} = 9.82 \text{ V}$$

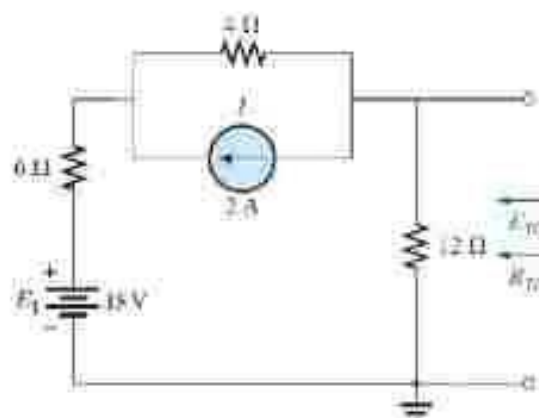


FIG. 9.55

Establishing the terminals of interest for the network of Fig. 9.54.

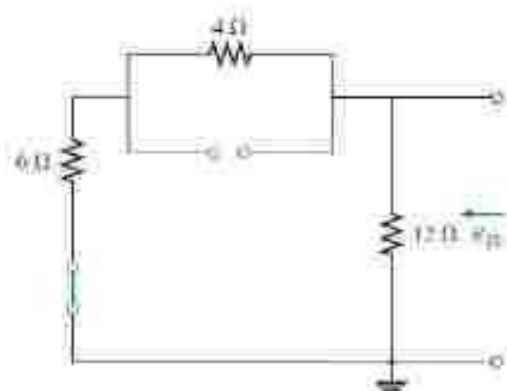
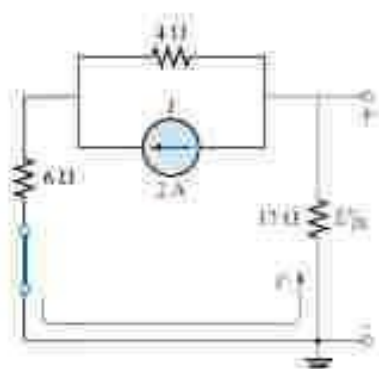
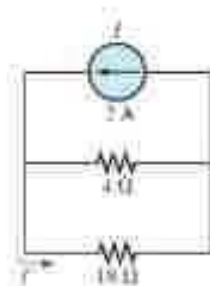


FIG. 9.56

Determining R_{TH} .



(a)



(b)

FIG. 9.58

Determining the contribution of I to E'_{TH} .

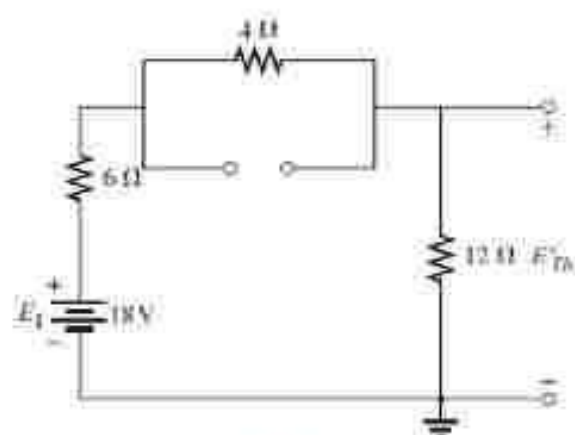


FIG. 9.57

Determining the contribution of E_1 to E'_{TH} .

The contribution due to the current source is determined using the network of Fig. 9.58(a) redrawn as shown in Fig. 9.58(b). Applying the current divider rule:

$$I' = \frac{4\Omega(2A)}{4\Omega + 18\Omega} = \frac{8}{22}A = 0.364A$$

$$\text{and } E'_{TH} = -I'(12\Omega) = -(0.364A)(12\Omega) = -4.37V$$

$$\text{so that } E_{TH} = E'_{TH} + E'_{TH} = 0.82V - 4.37V = -5.45V$$

b. The reconstructed network is shown in Fig. 9.59.

c. Using the voltage divider rule:

$$V_a = \frac{8\Omega(5.45V + 16V)}{5.45\Omega + 8\Omega} = \frac{8(21.45)}{13.45}V = \frac{171.6}{13.45}V = 12.76V$$

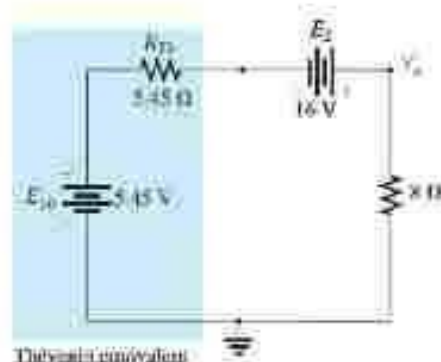


FIG. 9.59

Applying the Thévenin equivalent network to the network of Fig. 9.54.

Instead of using the superposition theorem, the current source could first have been converted to a voltage source and the series elements combined to determine the Thévenin voltage. In any event both approaches would have yielded the same results.

9.4 NORTON'S THEOREM

Any two-terminal linear bilateral dc network can be replaced by an equivalent circuit consisting of a current source and a parallel resistor, as shown in Fig. 9.65.

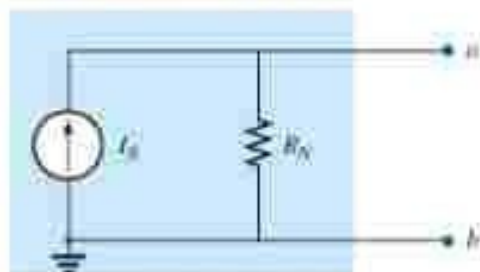


FIG. 9.65

Norton equivalent circuit.



FIG. 9.64

Edward L. Norton.

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The discussion of Thévenin's theorem with respect to the equivalent circuit can also be applied to the Norton equivalent circuit. The steps leading to the proper values of I_N and R_N are now listed.

Norton's Theorem Procedure

Preliminary:

1. Remove that portion of the network across which the Norton equivalent circuit is found.
2. Mark the terminals of the remaining two-terminal network.

R_N :

3. Calculate R_N by first setting all sources to zero (voltage sources are replaced with short circuits and current sources with open circuits) and then finding the resultant resistance between the two marked terminals. (If the internal resistance of the voltage and/or current sources is included in the original network, it must remain when the sources are set to zero.) Since $R_N = R_{Th}$, the procedure and value obtained using the approach described for Thévenin's theorem will determine the proper value of R_N .

I_N :

4. Calculate I_N by first returning all sources to their original position and then finding the short-circuit current between the marked terminals. It is the same current that would be measured by an ammeter placed between the marked terminals.

Conclusion:

5. Draw the Norton equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the equivalent circuit.

The Norton and Thévenin equivalent circuits can also be found from each other by using the source transformation discussed earlier in this chapter and reproduced in Fig. 9.66.

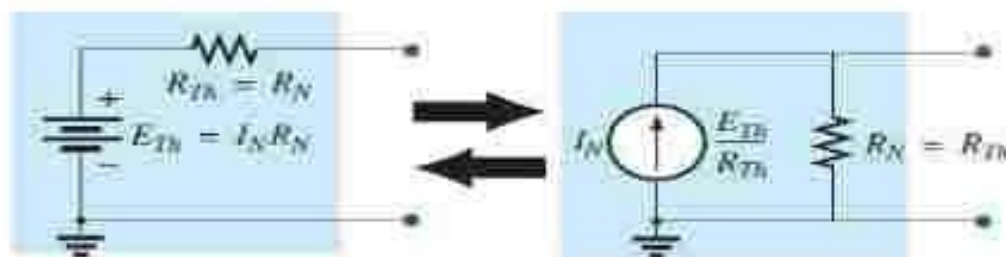


FIG. 9.66

Converting between Thévenin and Norton equivalent circuits.

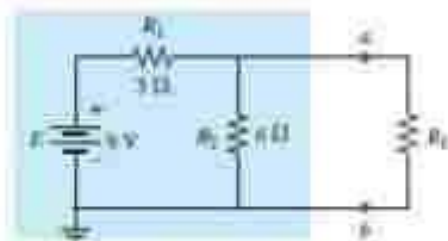


FIG. 9.67
Example 9.12

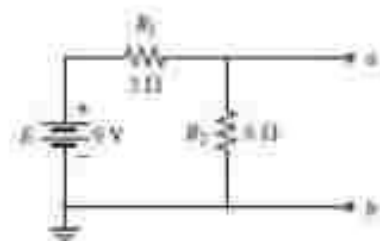


FIG. 9.68
Identifying the terminals of particular interest for the network in Fig. 9.67.

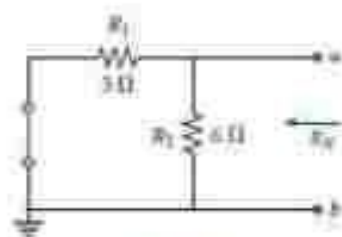


FIG. 9.69
Determining I_N for the network in Fig. 9.68.

EXAMPLE 9.12 Find the Norton equivalent circuit for the network in the shaded area in Fig. 9.67.

Solution:

Steps 1 and 2: See Fig. 9.68.

Step 3: See Fig. 9.69, and

$$R_N = R_1 \parallel R_2 = 3\Omega \parallel 6\Omega = \frac{(3\Omega)(6\Omega)}{3\Omega + 6\Omega} = \frac{18\Omega}{9} = 2\Omega$$

Step 4: See Fig. 9.70, which clearly indicates that the short-circuit connection between terminals a and b is in parallel with R_2 and eliminates its effect. I_N is therefore the same as through R_1 , and the full battery voltage appears across R_1 since

$$V_2 = I_2 R_2 = (0)6\Omega = 0\text{ V}$$

Therefore,

$$I_N = \frac{E}{R_1} = \frac{9\text{ V}}{3\Omega} = 3\text{ A}$$

Step 5: See Fig. 9.71. This circuit is the same as the first one considered in the development of Thévenin's theorem. A simple conversion indicates that the Thévenin circuits are, in fact, the same (Fig. 9.72).

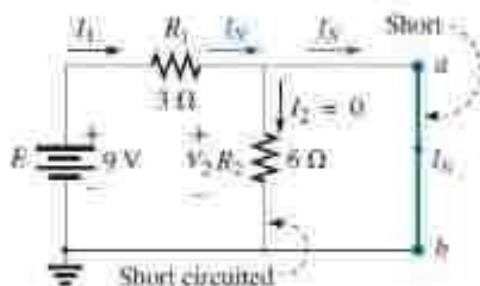


FIG. 9.70
Determining I_N for the network in Fig. 9.68.

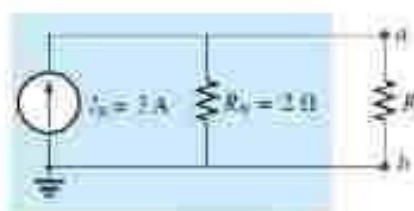


FIG. 9.71
Substituting the Norton equivalent circuit for the network external to the resistor R_L in Fig. 9.67.

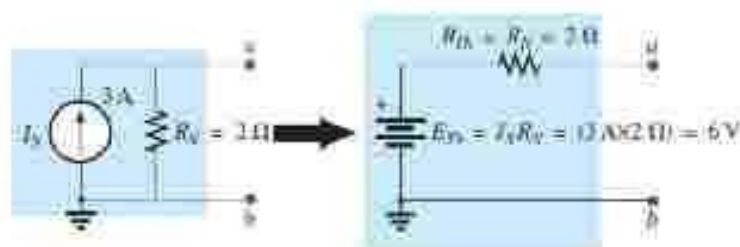


FIG. 9.72
Converting the Norton equivalent circuit in Fig. 9.71 to a Thévenin equivalent circuit.

EXAMPLE 9.13 Find the Norton equivalent circuit for the network external to the 9Ω resistor in Fig. 9.73.

Solution:

Steps 1 and 2: See Fig. 9.74.

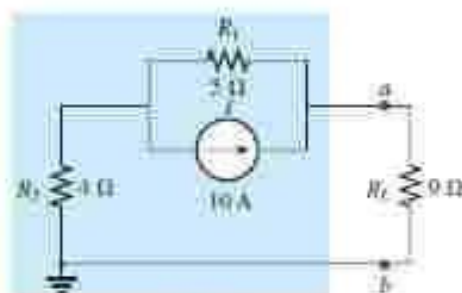


FIG. 9.73
Example 9.13.

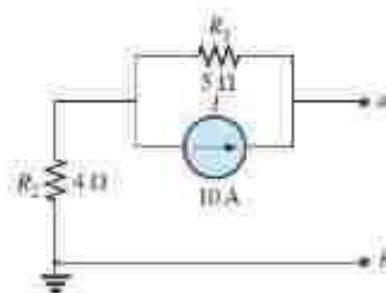


FIG. 9.74
Identifying the terminals of
particular interest for the network in
Fig. 9.73.

Step 3: See Fig. 9.75, and

$$R_N = R_1 + R_2 = 5 \Omega + 4 \Omega = 9 \Omega$$

Step 4: As shown in Fig. 9.76, the Norton current is the same as the current through the 4 Ω resistor. Applying the current divider rule gives

$$I_N = \frac{R_1 I}{R_1 + R_2} = \frac{(5 \Omega)(10 \text{ A})}{5 \Omega + 4 \Omega} = \frac{50 \text{ A}}{9} = 5.56 \text{ A}$$

Step 5: See Fig. 9.77.

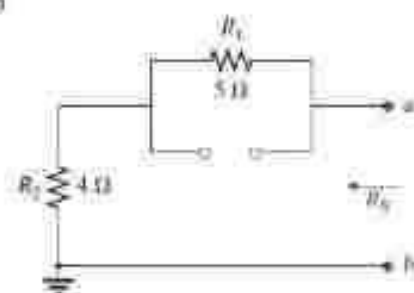


FIG. 9.75
Determining R_N for the network in
Fig. 9.74.

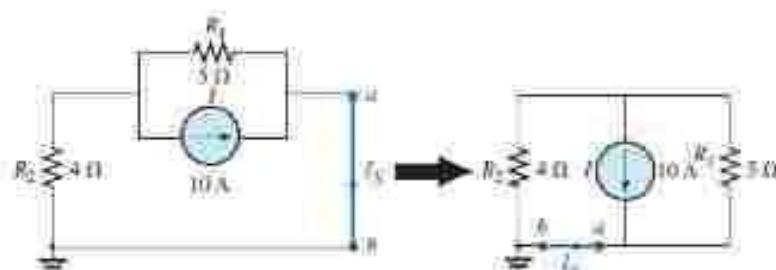


FIG. 9.76
Determining I_N for the network in Fig. 9.74.

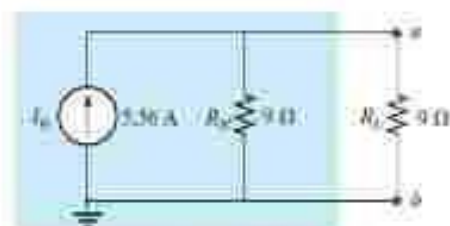


FIG. 9.77
Substituting the Norton equivalent circuit for the
network external in the resistor R_L in Fig. 9.73.

EXAMPLE 9.14 (Two sources) Find the Norton equivalent circuit for the portion of the network to the left of $a-b$ in Fig. 9.78.

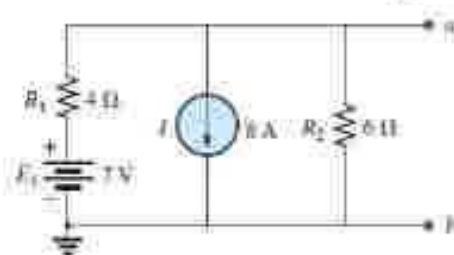


FIG. 9.78
Identifying the terminals of particular interest
for the network in Fig. 9.78.

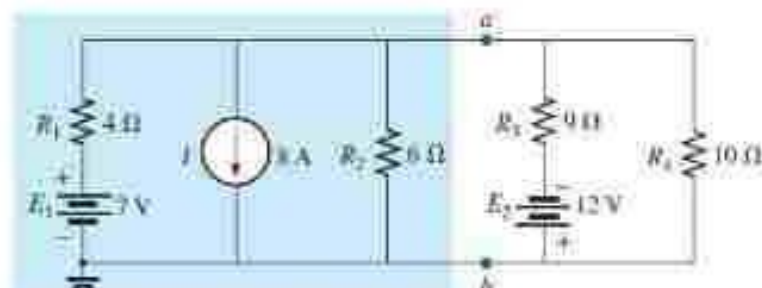


FIG. 9.78
Example 9.14.

Solution:

Steps 1 and 2: See Fig. 9.79.

Step 3: See Fig. 9.80, and

$$R_N = R_1 \parallel R_2 = 4 \Omega \parallel 6 \Omega = \frac{(4 \Omega)(6 \Omega)}{4 \Omega + 6 \Omega} = \frac{24 \Omega}{10} = 2.4 \Omega$$

Step 4: (Using superposition) For the 7 V battery (Fig. 9.81),

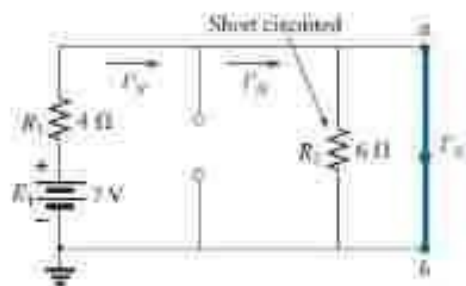


FIG. 9.81

Determining the contribution to I_x from the voltage source E_1 .

$$I_1 = \frac{E_1}{R_1} = \frac{7\text{ V}}{4\ \Omega} = 1.75\text{ A}$$

For the 8 A source (Fig. 9.82), we find that both R_1 and R_2 have been "short circuited" by the direct connection between a and b , and

$$I_2 = I = 8\text{ A}$$

The result is

$$I_x = I_2 - I_1 = 8\text{ A} - 1.75\text{ A} = 6.25\text{ A}$$

Step 5: See Fig. 9.83.

9.5 MAXIMUM POWER TRANSFER

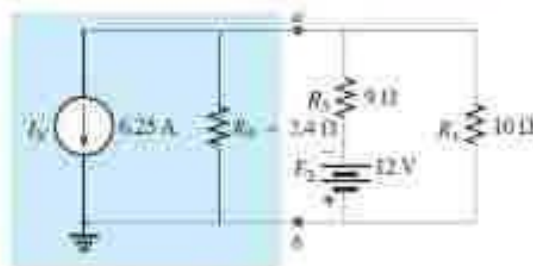


FIG. 9.83

Substituting the Norton equivalent circuit for the network to the left of terminals a – b in Fig. 9.78.

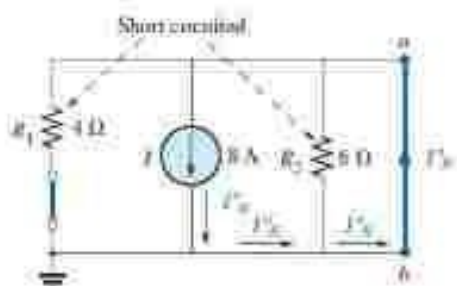


FIG. 9.82

Determining the contribution to I_x from the current source I .

THEOREM

When designing a circuit, it is often important to be able to answer one of the following questions:

What load should be applied to a system to ensure that the load is receiving maximum power from the system?

Conversely:

For a particular load, what conditions should be imposed on the source to ensure that it will deliver the maximum power available?

Even if a load cannot be set at the value that would result in maximum power transfer, it is often helpful to have some idea of the value that will draw maximum power so that you can compare it to the load at hand. For instance, if a design calls for a load of $100\ \Omega$, to ensure that the load receives maximum power, using a resistor of $1\ \Omega$ or $1\text{ k}\Omega$ results in a power transfer that is much less than the maximum possible.

However, using a load of $82\ \Omega$ or $120\ \Omega$ probably results in a fairly good level of power transfer. Fortunately, the process of finding the load that will receive maximum power from a particular system is quite straightforward due to the **maximum power transfer theorem**, which states the following:

A load will receive maximum power from a network when its resistance is exactly equal to the Thévenin resistance of the network applied to the load. That is,

$$R_L = R_{Th} \quad (9.2)$$

In other words, for the Thévenin equivalent circuit in Fig. 9.84, when the load is set equal to the Thévenin resistance, the load will receive maximum power from the network.

Using Fig. 9.84 with $R_L = R_{Th}$, we can determine the maximum power delivered to the load by first finding the current:

$$I_L = \frac{E_{Th}}{R_{Th} + R_L} = \frac{E_{Th}}{R_{Th} + R_{Th}} = \frac{E_{Th}}{2R_{Th}}$$

Then we substitute into the power equation:

$$P_L = I_L^2 R_L = \left(\frac{E_{Th}}{2R_{Th}} \right)^2 (R_{Th}) = \frac{E_{Th}^2 R_{Th}}{4R_{Th}^2}$$



FIG. 9.84

Defining the conditions for maximum power to a load using the Thévenin equivalent circuit.

and

$$P_{L_{\max}} = \frac{E_{Th}^2}{4R_{Th}} \quad (9.3)$$

The total power delivered by a supply such as E_{Th} is absorbed by both the Thévenin equivalent resistance and the load resistance. Any power delivered by the source that does not get to the load is lost to the Thévenin resistance.

Under maximum power conditions, only half the power delivered by the source gets to the load. Now, that sounds disastrous, but remember that we are starting out with a fixed Thévenin voltage and resistance, and the above simply tells us that we must make the two resistance levels equal if we want maximum power to the load. On an efficiency basis, we are working at only a 50% level, but we are content because we are getting maximum power out of our system.

The dc operating efficiency is defined as the ratio of the power delivered to the load (P_L) to the power delivered by the source (P_s). That is,

$$\eta\% = \frac{P_L}{P_s} \times 100\% \quad (9.4)$$

For the situation where $R_L = R_{Th}$,

$$\begin{aligned} \eta\% &= \frac{I_L^2 R_L}{I_L^2 R_T} \times 100\% = \frac{R_L}{R_T} \times 100\% = \frac{R_{Th}}{R_{Th} + R_{Th}} \times 100\% \\ &= \frac{R_{Th}}{2R_{Th}} \times 100\% = \frac{1}{2} \times 100\% = 50\% \end{aligned}$$

If efficiency is the overriding factor, then the load should be much larger than the internal resistance of the supply. If maximum power transfer is desired and efficiency less of a concern, then the conditions dictated by the maximum power transfer theorem should be applied.

A relatively low efficiency of 50% can be tolerated in situations where power levels are relatively low, such as in a wide variety of electronic systems, where maximum power transfer for the given system is usually more important. However, when large power levels are involved, such as at generating plants, efficiencies of 50% cannot be tolerated. In fact, a great deal of expense and research is dedicated to raising power generating and transmission efficiencies a few percentage points. Raising an efficiency level of a 10 MkW power plant from 94% to 95% (a 1% increase) can save 0.1 MkW, or 100 million watts, of power—an enormous saving. In all of the above discussions, the effect of changing the load was discussed for a fixed Thévenin resistance. Looking at the situation from a different viewpoint, we can say

if the load resistance is fixed and does not match the applied Thévenin equivalent resistance, then some effort should be made (if possible) to redesign the system so that the Thévenin equivalent resistance is closer to the fixed applied load.

In other words, if a designer faces a situation where the load resistance is fixed, he or she should investigate whether the supply section should be replaced or redesigned to create a closer match of resistance levels to produce higher levels of power to the load.

For the Norton equivalent circuit in Fig. 9.90, maximum power will be delivered to the load when,

$$R_L = R_N \quad (9.5)$$

This result [Eq. (9.5)] will be used to its fullest advantage in the analysis of transistor networks, where the most frequently applied transistor circuit model uses a current source rather than a voltage source.

For the Norton circuit in Fig. 9.90,

$$P_{L_{\max}} = \frac{I_N^2 R_N}{4} \quad (\text{W}) \quad (9.6)$$

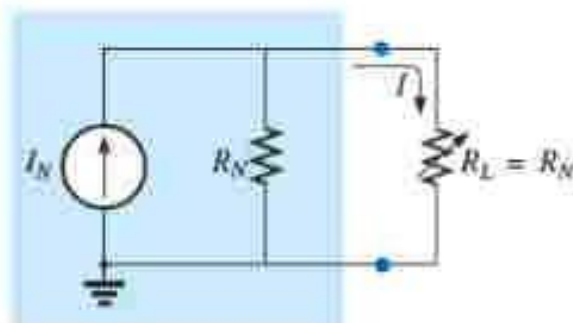


FIG. 9.90

Defining the conditions for maximum power to a load using the Norton equivalent circuit.

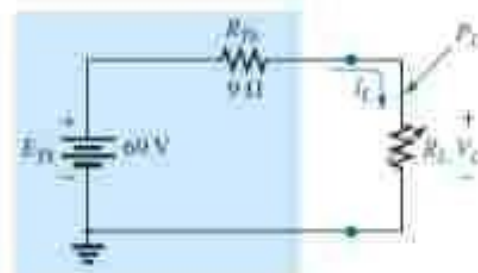


FIG. 9.85

Thevenin equivalent network to be used to validate the maximum power transfer theorem.

To demonstrate that maximum power is indeed transferred to the load under the conditions defined above, consider the Thévenin equivalent circuit in Fig. 9.85. Before getting into detail, however, if you were to guess what value of R_L would result in maximum power transfer to R_L , you might think that the smaller the value of R_L , the better it is because the current reaches a maximum when it is squared in the power equation. The problem is, however, that in the equation $PL = I^2$

LRL , the load resistance is a multiplier. As it gets smaller, it forms a smaller product. Then again, you might suggest larger values of R_L because the output voltage increases, and power is determined by $PL = V^2 / L / RL$. This time, however, the load resistance is in the denominator of the equation and causes the resulting power to decrease. A balance must obviously be made between the load resistance and the resulting current or voltage. The following discussion shows that *maximum power transfer occurs when the load voltage and current are one-half their maximum possible values.*

For the circuit in Fig. 9.85, the current through the load is determined by

$$I_L = \frac{E_{TH}}{R_{TH} + R_L} = \frac{60 \text{ V}}{9 \Omega + R_L}$$

The voltage is determined by

$$V_L = \frac{R_L E_{TH}}{R_L + R_{TH}} = \frac{R_L (60 \text{ V})}{R_L + R_{TH}}$$

and the power by

$$P_L = I_L^2 R_L = \left(\frac{60 \text{ V}}{9 \Omega + R_L} \right)^2 (R_L) = \frac{3600 R_L}{(9 \Omega + R_L)^2}$$

If we tabulate the three quantities versus a range of values for R_L from 0.1Ω to 30Ω , we obtain the results appearing in Table 9.1. Note in particular that when R_L is equal to the Thévenin resistance of 9Ω , the

power has a maximum value of 100 W, the current is 3.33 A, or one-half its maximum value of 6.67 A (as would result with a short circuit across the output terminals), and the voltage across the load is 30 V, or one-half its maximum value of 60 V (as would result with an open circuit across its output terminals). As you can see, there is no question that maximum power is transferred to the load when the load equals the Thévenin value.

The power to the load versus the range of resistor values is provided in Fig. 9.86. Note in particular that for values of load resistance less than the Thévenin value, the change is dramatic as it approaches the peak value. However, for values greater than the Thévenin value, the drop is a great deal more gradual. This is important because it tells us the following:

If the load applied is less than the Thévenin resistance, the power to the load will drop off rapidly as it gets smaller. However, if the applied load is greater than the Thévenin resistance, the power to the load will not drop off as rapidly as it increases.

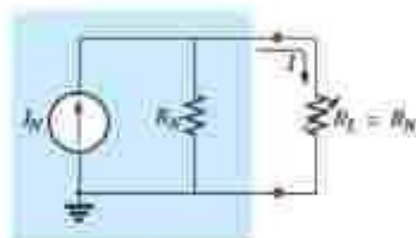


FIG. 9.90

Defining the conditions for maximum power to a load using the Norton equivalent circuit.

In all of the above discussions, the effect of changing the load was discussed for a fixed Thévenin resistance. Looking at the situation from a different viewpoint, we can say

if the load resistance is fixed and does not match the applied Thévenin equivalent resistance, then some effort should be made (if possible) to redesign the system so that the Thévenin equivalent resistance is closer to the fixed applied load.

In other words, if a designer faces a situation where the load resistance is fixed, he or she should investigate whether the supply section should be replaced or redesigned to create a closer match of resistance levels to produce higher levels of power to the load.

For the Norton equivalent circuit in Fig. 9.90, maximum power will be delivered to the load when

$$R_L = R_N \quad (9.5)$$

This result [Eq. (9.5)] will be used to its fullest advantage in the analysis of transistor networks, where the most frequently applied transistor circuit model uses a current source rather than a voltage source.

For the Norton circuit in Fig. 9.90,

$$P_{\text{max}} = \frac{I_N^2 R_N}{4} \quad (\text{W}) \quad (9.6)$$

EXAMPLE 9.15 A dc generator, battery, and laboratory supply are connected to resistive load R_L in Fig. 9.91.

- For each, determine the value of R_L for maximum power transfer to R_L .
- Under maximum power conditions, what are the current level and the power to the load for each configuration?
- What is the efficiency of operation for each supply in part (b)?
- If a load of 1 k Ω were applied to the laboratory supply, what would the power delivered to the load be? Compare your answer to the level of part (b). What is the level of efficiency?
- For each supply, determine the value of R_L for 75% efficiency.

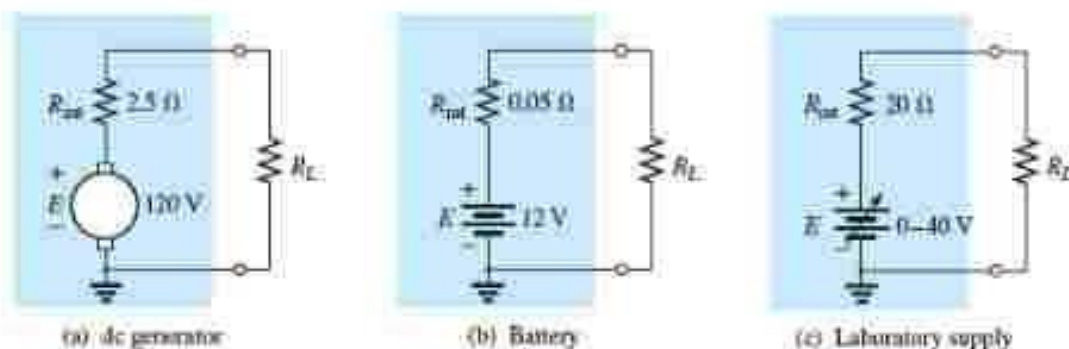


FIG. 9.91
Example 9.15.

Solutions:

a. For the dc generator,

$$R_L = R_{Th} = R_{int} = 2.5 \, \Omega$$

For the 12 V car battery,

$$R_L = R_{Th} = R_{int} = 0.05 \, \Omega$$

For the dc laboratory supply,

$$R_L = R_{Th} = R_{int} = 20 \, \Omega$$

b. For the dc generator,

$$P_{L_{max}} = \frac{E_{Th}^2}{4R_{Th}} = \frac{E^2}{4R_{int}} = \frac{(120 \, \text{V})^2}{4(2.5 \, \Omega)} = 1.44 \, \text{kW}$$

For the 12 V car battery,

$$P_{L_{max}} = \frac{E_{Th}^2}{4R_{Th}} = \frac{E^2}{4R_{int}} = \frac{(12 \, \text{V})^2}{4(0.05 \, \Omega)} = 720 \, \text{W}$$

For the dc laboratory supply,

$$P_{L_{max}} = \frac{E_{Th}^2}{4R_{Th}} = \frac{E^2}{4R_{int}} = \frac{(40 \, \text{V})^2}{4(20 \, \Omega)} = 20 \, \text{W}$$

c. They are all operating under a 50% efficiency level because $R_L = R_{Th}$.

d. The power to the load is determined as follows:

$$I_L = \frac{E}{R_{int} + R_L} = \frac{40 \, \text{V}}{20 \, \Omega + 1000 \, \Omega} = \frac{40 \, \text{V}}{1020 \, \Omega} = 39.22 \, \text{mA}$$

and $P_L = I_L^2 R_L = (39.22 \, \text{mA})^2 (1000 \, \Omega) = 1.54 \, \text{W}$

The power level is significantly less than the 20 W achieved in part (b). The efficiency level is

$$\begin{aligned} \eta\% &= \frac{P_L}{P_s} \times 100\% = \frac{1.54 \, \text{W}}{EI_s} \times 100\% = \frac{1.54 \, \text{W}}{(40 \, \text{V})(39.22 \, \text{mA})} \times 100\% \\ &= \frac{1.54 \, \text{W}}{1.57 \, \text{W}} \times 100\% = 98.09\% \end{aligned}$$

which is markedly higher than achieved under maximum power conditions—albeit at the expense of the power level.

c. For the dc generator,

$$\eta = \frac{P_r}{P_s} = \frac{R_L}{R_{Th} + R_L} \quad (\eta \text{ in decimal form})$$

and
$$\eta = \frac{R_L}{R_{Th} + R_L}$$

$$\eta(R_{Th} + R_L) = R_L$$

$$\eta R_{Th} + \eta R_L = R_L$$

$$R_L(1 - \eta) = \eta R_{Th}$$

and

$$R_L = \frac{\eta R_{Th}}{1 - \eta} \quad (9.7)$$

$$R_L = \frac{0.75(2.5 \, \Omega)}{1 - 0.75} = 7.5 \, \Omega$$

For the battery,

$$R_L = \frac{0.75(0.05 \, \Omega)}{1 - 0.75} = 0.15 \, \Omega$$

For the laboratory supply,

$$R_L = \frac{0.75(20 \, \Omega)}{1 - 0.75} = 60 \, \Omega$$



FIG. 9.32
Example 9.16

EXAMPLE 9.16 The analysis of a transistor network resulted in the reduced equivalent in Fig. 9.32.

- Find the load resistance that will result in maximum power transfer to the load, and find the maximum power delivered.
- If the load were changed to 68 kΩ, would you expect a fairly high level of power transfer to the load based on the results of part (a)? What would the new power level be? Is your initial assumption verified?
- If the load were changed to 8.2 kΩ, would you expect a fairly high level of power transfer to the load based on the results of part (a)? What would the new power level be? Is your initial assumption verified?

Solutions:

- Replacing the current source by an open-circuit equivalent results in

$$R_{Th} = R_s = 60 \, \text{k}\Omega$$

Restoring the current source and finding the open-circuit voltage at the output terminals results in

$$E_{Th} = V_{oc} = IR_s = (10 \, \text{mA})(60 \, \text{k}\Omega) = 400 \, \text{V}$$

For maximum power transfer to the load,

$$R_L = R_{Th} = 60 \, \text{k}\Omega$$

with a maximum power level of

$$P_{\max} = \frac{E_{Th}^2}{4R_{Th}} = \frac{(400 \, \text{V})^2}{4(60 \, \text{k}\Omega)} = 1 \, \text{W}$$

- b. Yes, because the 68 k Ω load is greater (note Fig. 9.86) than the 40 k Ω load, but relatively close in magnitude.

$$I_L = \frac{E_{Th}}{R_{Th} + R_L} = \frac{400 \text{ V}}{40 \text{ k}\Omega + 68 \text{ k}\Omega} = \frac{400}{108 \text{ k}\Omega} \approx 3.7 \text{ mA}$$

$$P_L = I_L^2 R_L = (3.7 \text{ mA})^2 (68 \text{ k}\Omega) \approx 0.93 \text{ W}$$

Yes, the power level of 0.93 W compared to the 1 W level of part (a) verifies the assumption.

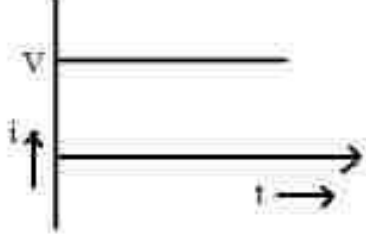
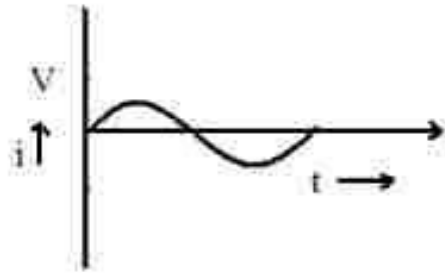
- c. No, 8.2 k Ω is quite a bit less (note Fig. 9.86) than the 40 k Ω value.

$$I_L = \frac{E_{Th}}{R_{Th} + R_L} = \frac{400 \text{ V}}{40 \text{ k}\Omega + 8.2 \text{ k}\Omega} = \frac{400 \text{ V}}{48.2 \text{ k}\Omega} \approx 8.3 \text{ mA}$$

$$P_L = I_L^2 R_L = (8.3 \text{ mA})^2 (8.2 \text{ k}\Omega) \approx 0.57 \text{ W}$$

Yes, the power level of 0.57 W compared to the 1 W level of part (a) verifies the assumption.

CHAPTER-05
AC CIRCUIT AND RESONANCE

<u>Direct Current</u>	<u>Alternating Current</u>
	
<p>(1) D.C. always flow in one direction and whose magnitude remains constant.</p> <p>(2) High cost of production.</p> <p>(3) It is not possible by D.C. Because D.C. is dangerous to the transformer.</p> <p>(4) Its transmission cost is too high.</p>	<p>(1) A.C. is one which reverse periodically in direction and whose magnitude undergoes a definite cycle changes in definite intervals of time.</p> <p>(2) Low cost of production</p> <p>(3) By using transformer A.C. voltage can be decreased or increased.</p> <p>(4) A.C. can be transmitted to a long distance economically.</p>

Definition of A.C. terms :-

Cycle : It is one complete set of +ve and -ve values of alternating quality spread over 360° or 2π radian.

Time Period : It is defined as the time required to complete one cycle.

Frequency : It is defined as the reciprocal of time period. i.e. $f = 1/T$

Or

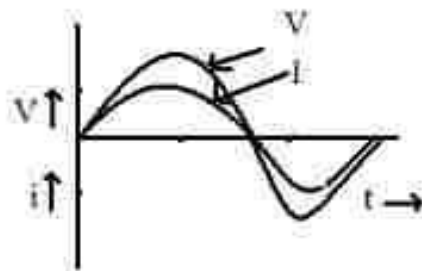
It is defined as the number of cycles completed per second.

Amplitude : It is defined as the maximum value of either +ve half cycle or -ve half cycle.

Phase : It is defined as the angular displacement between two halves is zero.

OR

Two alternating quantity are in phase when each pass through their zero value at the same instant and also attain their maximum value at the same instant in a given cycle.



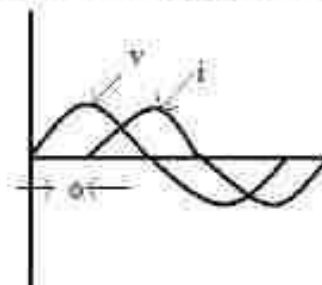
$$V = V_m \sin \omega t$$

$$i = I_m \sin \omega t$$

Phase Difference :- It is defined as the angular displacement between two alternating quantities.

OR

If the angular displacement between two waves are not zero, then that is known as phase difference, i.e. at a particular time they attain unequal distance.



OR

Two quantities are out of phase if they reach their maximum value or minimum value at different times but always have an equal phase angle between them.

Here $V = V_m \sin \omega t$

$$i = I_m \sin (\omega t - \phi)$$

In this case current lags voltage by an angle ' ϕ '.

Phasor Diagram :

Generation of Alternating emf :-

Consider a rectangular coil of ' N ' turns, area of cross-section is ' A ' m^2 is placed in x -axis in an uniform magnetic field of maximum flux density $B_m \text{ web/m}^2$. The coil is rotating in the magnetic field with a velocity of ω radian / second. At time $t = 0$, the coil is in x -axis. After interval of time ' dt ' second the coil make rotating in anti-clockwise direction and makes an angle ' θ ' with x -direction. The perpendicular component of the magnetic field is $\phi = \phi_m \cos \omega t$

According to Faraday's Laws of electro-magnetic Induction

$$\begin{aligned}
 e &= -N \frac{d\phi}{dt} \\
 &= -N \frac{d}{dt} (\phi_m \cos \omega t) \\
 &= -N (-\phi_m \omega \sin \omega t) \\
 &= N \omega \phi_m \sin \omega t \\
 &= 2\pi f N \phi_m \sin \omega t \quad (\because \omega = 2\pi f) \\
 &= 2\pi f N B_m A \sin \omega t \\
 e &= E_m \sin \omega t
 \end{aligned}$$

Where $E_m = 2\pi f N B_m A$

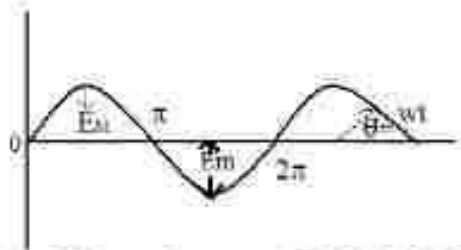
$f \rightarrow$ frequency in Hz

$B_m \rightarrow$ Maximum flux density in Wb/m²

Now when θ or $\omega t = 90^\circ$

$$e = E_m$$

i.e. $E_m = 2\pi f N B_m A$



Root Mean Square (R.M.S) Value \rightarrow

The r.m.s. value of an a.c. is defined by that steady (d.c.) current which when flowing through a given circuit for a given time produces same heat as produced by the alternating current when flowing through the same circuit for the same time.

Sinusoidal alternating current is

$$i = I_m \sin \omega t = I_m \sin \theta$$

The mean of squares of the instantaneous values of current over one complete cycle

$$= \int_0^{2\pi} \frac{i^2 d\theta}{(2\pi - 0)}$$

The square root of this value is

$$\begin{aligned}
 &= \sqrt{\int_0^{2\pi} \frac{i^2 d\theta}{2\pi}} \\
 &= \sqrt{\int_0^{2\pi} \frac{(I_m \sin \theta)^2 d\theta}{2\pi}}
 \end{aligned}$$

$$\begin{aligned}
&= \sqrt{\frac{I_m^2}{2\pi} \int_0^{2\pi} \sin^2 \theta d\theta} \\
&= \sqrt{\frac{I_m^2}{2\pi} \int_0^{2\pi} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta} \\
&= \sqrt{\frac{I_m^2}{4\pi} \int_0^{2\pi} (1 - \cos 2\theta) d\theta} \\
&= \sqrt{\frac{I_m^2}{4\pi} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi}} \\
&= \sqrt{\frac{I_m^2}{4\pi} \int_0^{2\pi} \left(2\pi - \frac{\sin 4\pi}{2} \right) d\theta} \\
&= \sqrt{\frac{I_m^2}{4\pi} \int_0^{2\pi} (2\pi - 0) d\theta} \\
&= \sqrt{\frac{I_m^2}{2}} = \frac{I_m}{\sqrt{2}} \\
I_{rms} &= \frac{I_m}{\sqrt{2}} = 0.707 I_m
\end{aligned}$$

Average Value :→

The average value of an alternating current is expressed by that steady current (d.c.) which transfers across any circuit the same charge as it transferred by that alternating current during the same time.

The equation of the alternating current is $i = I_m \sin \theta$

$$\begin{aligned}
I_m &= \int_0^{\pi} \frac{i d\theta}{\pi} \\
&= \int_0^{\pi} \frac{I_m \sin \theta}{\pi} d\theta = \frac{I_m}{\pi} \int_0^{\pi} \sin \theta d\theta \\
&= \frac{I_m}{\pi} [-\cos \theta]_0^{\pi} = \frac{I_m}{\pi} [-\cos \pi - (-\cos 0)] \\
&= \frac{I_m}{\pi} [1 - 0(-1)] \\
I_m &= \frac{2I_m}{\pi} \\
I_m &= \frac{2 \times \text{Maximum Current}}{\pi} \\
\text{Hence, } I_m &= 0.637 I_m
\end{aligned}$$

The average value over a complete cycle is zero.

Amplitude factor/ Peak factor/ Crest factor :- It is defined as the ratio of maximum value to r.m.s value.

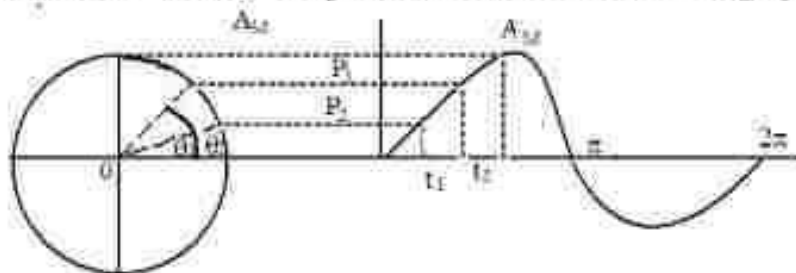
$$K_o = \frac{\text{Maximum Value}}{\text{R.M.S Value}} = \frac{I_m}{\frac{I_m}{\sqrt{2}}} = \sqrt{2} = 1.414$$

Form factor :- It is defined as the ratio of r.m.s value to average value.

$$K_f = \frac{\text{r.m.s Value}}{\text{Average Value}} = \frac{0.707 I_m}{0.637 I_m} = \sqrt{2} = 1.414$$

$$K_f = 1.11$$

Phasor or Vector Representation of Alternating Quantity :→



An alternating current or voltage, (quantity) in a vector quantity which has magnitude as well as direction. Let the alternating value of current be represented by the equation $e = E_m \sin \omega t$. The projection of E_m on Y-axis at any instant gives the instantaneous value of alternating current. Since the instantaneous values are continuously changing, so they are represented by a rotating vector or phasor. A phasor is a vector rotating at a constant angular velocity

$$\text{At } t_1, e_1 = E_m \sin \omega t_1$$

$$\text{At } t_2, e_2 = E_m \sin \omega t_2$$

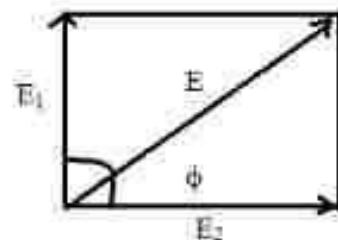
Addition of two alternating Current :→

$$\text{Let } e_1 = E_m \sin \omega t$$

$$e_2 = E_m \sin(\omega t - \phi)$$

The sum of two sine waves of the same frequency is another sine wave of same frequency but of a different maximum value and Phase.

$$e = \sqrt{e_1^2 + e_2^2 + 2e_1 e_2 \cos \phi}$$



Phasor Algebra :→

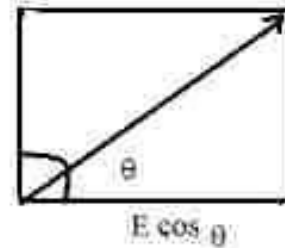
A vector quantity can be expressed in terms of

- (i) Rectangular or Cartesian form
- (ii) Trigonometric form
- (iii) Exponential form

(iv) Polar form

$$E = a + jb$$

$$= E(\cos \theta + j \sin \theta)$$



Where $a = E \cos \theta$ is the active part

$b = E \sin \theta$ is the reactive part

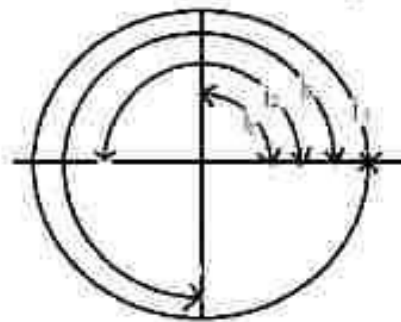
$$\theta = \tan^{-1} \left(\frac{b}{a} \right) = \text{Phase angle}$$

$$j = \sqrt{-1} (90^\circ)$$

$$j^2 = -1 (180^\circ)$$

$$j^3 = -j (270^\circ)$$

$$j^4 = 1 (360^\circ)$$



(i) Rectangular form :-

$$E = a + jb$$

$$\tan \theta = b / a$$

(ii) Trigonometric form :-

$$E = E(\cos \theta + j \sin \theta)$$

(iii) Exponential form :-

$$E = E e^{j\theta}$$

(iv) Polar form :-

$$E = E \angle \theta \quad (E = \sqrt{a^2 + b^2})$$

Addition or Subtraction :-

$$E_1 = a_1 + jb_1$$

$$E_2 = a_2 + jb_2$$

$$E_1 \pm E_2 = (a_1 \pm a_2) \pm (b_1 \pm b_2)$$

$$\phi = \tan^{-1} \left(\frac{b_1 + b_2}{a_1 + a_2} \right)$$

Multiplication :-

$$E_1 \times E_2 = (a_1 + ja__1) \pm (a_2 + jb_2)$$

$$= (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + b_1 a_2)$$

$$\phi = \tan^{-1} \left(\frac{a_1 b_2 + b_1 a_2}{a_1 a_2 - b_1 b_2} \right)$$

$$E_1 = E_1 \angle \theta_1$$

$$E_2 = E_2 \angle \theta_2$$

$$E_1 \times E_2 = E_1 E_2 \angle \phi_1 + \phi_2$$

Division :-

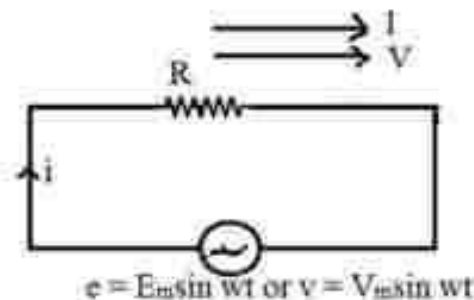
$$E_1 = E_1 \angle \theta_1$$

$$E_2 = E_2 \angle \theta_2$$

$$\frac{E_1}{E_2} = \frac{E_1 \angle \theta_1}{E_2 \angle \theta_2} = \frac{E_1}{E_2} \angle \theta_1 - \theta_2$$

A.C. through Pure Resistance :→

Let the resistance of R ohm is connected across to A.C supply of applied voltage



$$e = E_m \sin \omega t \text{ ----- (1)}$$

Let 'i' is the instantaneous current .

Here $e = iR$

$$\Rightarrow i = e/R$$

$$i = E_m \sin \omega t / R \text{ ----- (2)}$$

By comparing equation (1) and equation (2) we get alternating voltage and current in a pure resistive circuit are in phase

Instantaneous power is given by

$$P = ei$$

$$= E_m \sin \omega t \cdot I_m \sin \omega t$$

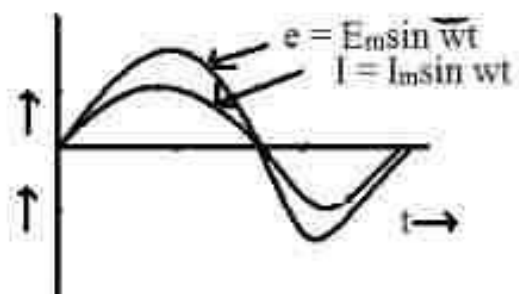
$$= E_m I_m \sin^2 \omega t$$

$$= \frac{E_m I_m}{2} \sin^2 \omega t$$

$$= \frac{E_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} (1 - \cos 2\omega t)$$

$$P = \frac{E_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} - \frac{E_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} \cos 2\omega t$$

$$\text{i.e. } P = \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} - \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} \cos 2\omega t$$



Where $\frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}}$ is called constant part of power.

$\frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} \cos 2\omega t$ is called fluctuating part of power.

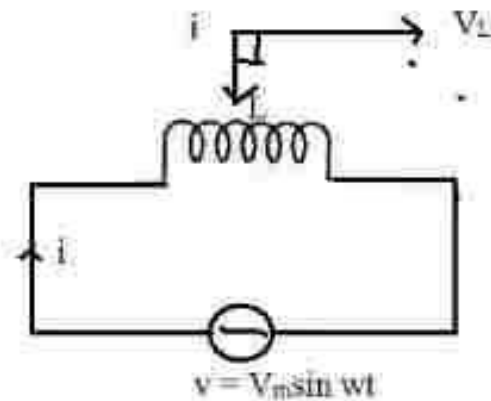
The fluctuating part $\frac{V_m I_m}{2} \cos 2\omega t$ of frequency double that of voltage and current waves.

Hence power for the whole cycle is $P = \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} = V_{rms} I_{rms}$

$$\Rightarrow P = VI \text{ watts}$$

A.C through Pure Inductance :→

Let inductance of 'L' henry is connected across the A.C. supply



$$v = V_m \sin \omega t \text{ ----- (1)}$$

According to Faraday's laws of electromagnetic induction the emf induced across the inductance

$$V = L \frac{di}{dt}$$

$\frac{di}{dt}$ is the rate of change of current

$$V_m \sin \omega t = L \frac{di}{dt}$$

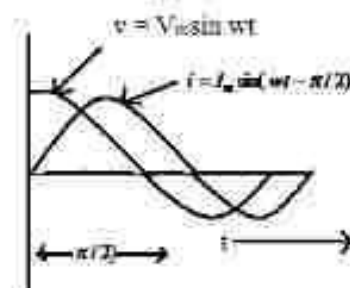
$$\frac{di}{dt} = \frac{V_m \sin \omega t}{L}$$

$$\Rightarrow di = \frac{V_m}{L} \sin \omega t \cdot dt$$

Integrating both sides,

$$\int di = \int \frac{V_m}{L} \sin \omega t \cdot dt$$

$$i = \frac{V_m}{L} \left(-\frac{\cos \omega t}{\omega} \right)$$



$$i = \frac{V_m \cos \omega t}{\omega L}$$

$$i = -\frac{V_m}{\omega L} \sin \omega t$$

$$i = -\frac{V_m}{\omega L} \sin \left(\omega t - \frac{\pi}{2} \right)$$

$$= -\frac{V_m}{X_L} \sin \left(\omega t - \frac{\pi}{2} \right) \quad [\because X_L = 2\pi fL = \omega L]$$

Maximum value of i is

$$I_m = \frac{V_m}{X_L} \text{ when } \sin \left(\omega t - \frac{\pi}{2} \right) \text{ is unity.}$$

Hence the equation of current becomes $i = I_m \sin(\omega t - \pi/2)$

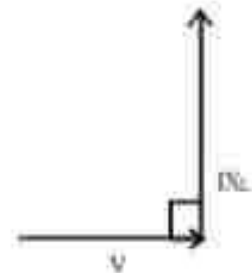
So we find that if applied voltage is represented by $v = V_m \sin \omega t$, then current flowing in a purely inductive circuit is given by

$$i = I_m \sin(\omega t - \pi/2)$$

Here current lags voltage by an angle $\pi/2$ Radian.

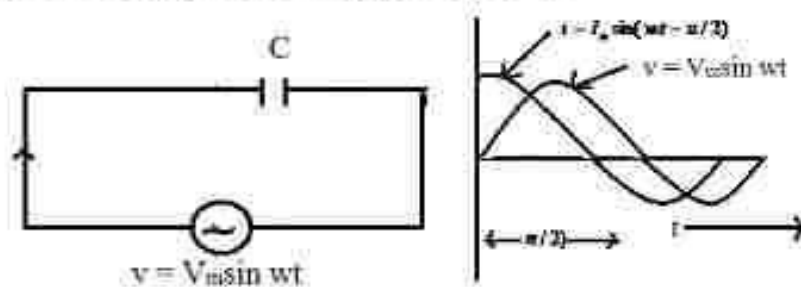
$$\begin{aligned} \text{Power factor} &= \cos \phi \\ &= \cos 90^\circ \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{Power Consumed} &= VI \cos \phi \\ &= VI \times 0 \\ &= 0 \end{aligned}$$



Hence, the power consumed by a purely Inductive circuit is zero.

A.C. Through Pure Capacitance : →



Let a capacitance of 'C' farad is connected across the A.C. supply of applied voltage

$$v = V_m \sin \omega t \text{ ----- (1)}$$

Let 'q' = charge on plates when p.d. between two plates of capacitor is 'v'

$$q = cv$$

$$q = cV_m \sin \omega t$$

$$\frac{dq}{dt} = \varepsilon \frac{d}{dt} (V_m \sin \omega t)$$

$$i = \varepsilon V_m \sin \omega t$$

$$= \omega \varepsilon V_m \cos \omega t$$

$$= \frac{V_m}{1/\omega \varepsilon} \cos \omega t$$

$$= \frac{V_m}{X_c} \cos \omega t$$

$$[\because X_c = \frac{1}{\omega \varepsilon} = \frac{1}{2\pi f \varepsilon} \text{ is known as capacitive reactance}$$

in ohm,]

$$= I_m \cos \omega t$$

$$= I_m \sin(\omega t + \pi/2)$$

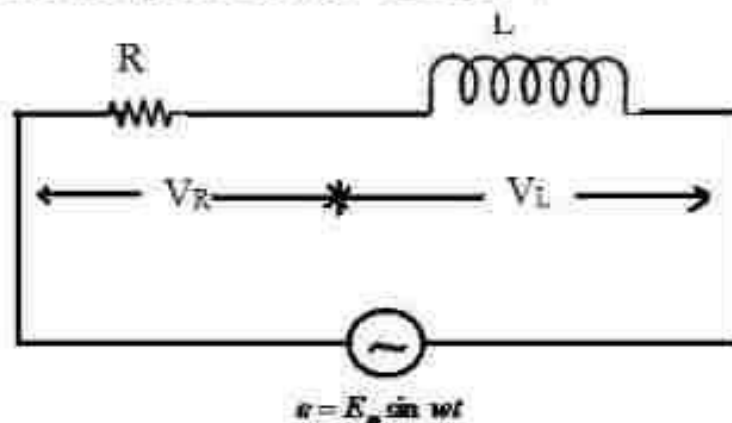
Here current leads the supply voltage by an angle $\pi/2$ radian.

$$\begin{aligned} \text{Power factor} &= \cos \phi \\ &= \cos 90^\circ = 0 \end{aligned}$$

$$\begin{aligned} \text{Power Consumed} &= VI \cos \phi \\ &= VI \times 0 = 0 \end{aligned}$$

The power consumed by a pure capacitive circuit is zero.

A.C. Through R-L Series Circuit : →



The resistance of R-ohm and inductance of L-henry are connected in series across the A.C. supply of applied voltage

$$e = E_m \sin \omega t \text{ ----- (1)}$$

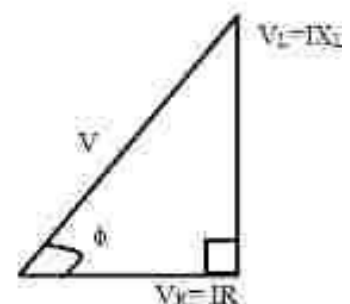
$$V = V_R + jV_L$$

$$= \sqrt{V_R^2 + V_L^2} \angle \phi = \tan^{-1} \left(\frac{X_L}{R} \right)$$

$$= \sqrt{(IR)^2 + (IX_L)^2} \angle \phi = \tan^{-1} \left(\frac{X_L}{R} \right)$$

$$= I \sqrt{R^2 + X_L^2} \angle \phi = \tan^{-1} \left(\frac{X_L}{R} \right)$$

$$V = IZ \angle \phi = \tan^{-1} \left(\frac{X_L}{R} \right)$$



Where $Z = \sqrt{R^2 + X_L^2}$

$= R + jX_L$ is known as impedance of R-L series Circuit.

$$I = \frac{V}{Z \angle \phi} = \frac{E_m \sin \omega t}{Z \angle \phi}$$

$$I = I_m \sin(\omega t - \phi)$$

Here current lags the supply voltage by an angle ϕ .

Power Factor :→ It is the cosine of the angle between the voltage and current.

OR

It is the ratio of active power to apparent power.

OR

It is the ratio of resistance to impedance.

Power :→

$$= v i$$

$$= V_m \sin \omega t \cdot I_m \sin(\omega t - \phi)$$

$$= V_m I_m \sin \omega t \cdot \sin(\omega t - \phi)$$

$$= \frac{1}{2} V_m I_m 2 \sin \omega t \cdot \sin(\omega t - \phi)$$

$$= \frac{1}{2} V_m I_m [\cos \phi - \cos 2(\omega t - \phi)]$$

Obviously the power consists of two parts:

(i) a constant part $\frac{1}{2} V_m I_m \cos \phi$ which contributes to real power.

(ii) a pulsating component $\frac{1}{2} V_m I_m \cos(2\omega t - \phi)$ which has a frequency twice that of the voltage and current. It does not contribute to actual power since its average value over a complete cycle is zero.

Hence average power consumed

$$= \frac{1}{2} V_m I_m \cos \phi$$

$$= \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} \cos \phi$$

$$= VI \cos \phi$$

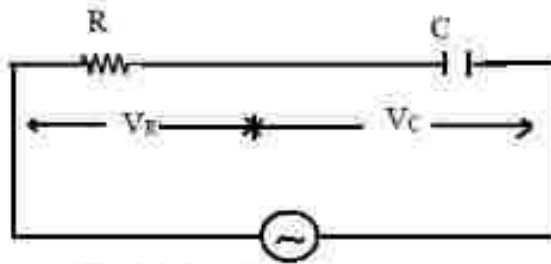
Where V & I represents the r.m.s value.

A.C. Through R-C Series Circuit : →

The resistance of 'R'-ohm and capacitance of 'C' farad is connected across the A.C. supply of applied voltage

$$e = E_m \sin \omega t$$

$$\text{-----}(1)$$



$$V = V_R + (-jV_C)$$

$$= IR + (-jIX_C)$$

$$= I(R - jX_C)$$

$$V = IZ$$

Where $Z = R - jX_C = \sqrt{R^2 + X_C^2}$ is known as impedance of R-C series Circuit.

$$Z = R - jX_C$$

$$= \sqrt{R^2 + X_C^2}$$

$$\angle -\phi = \tan^{-1} \left(\frac{X_C}{R} \right)$$

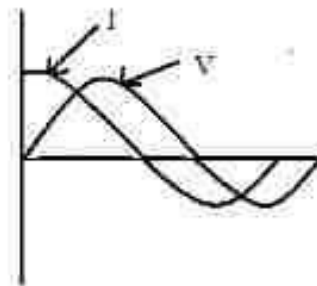
$$V = IZ \angle -\phi$$

$$\Rightarrow I = \frac{V}{Z \angle -\phi}$$

$$= \frac{E_m \sin \omega t}{Z \angle -\phi}$$

$$= \frac{E}{Z} \sin(\omega t + \phi)$$

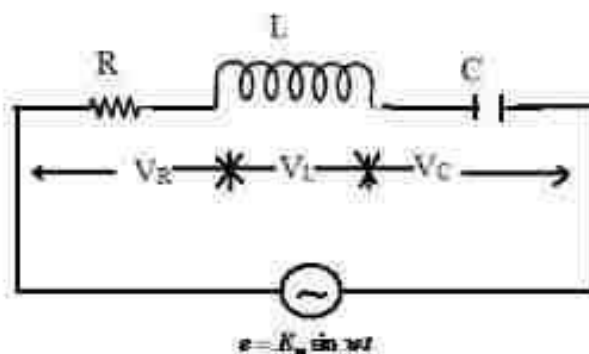
$$\Rightarrow I = I_m \sin(\omega t + \phi)$$



Here current leads the supply voltage by an angle ' ϕ '.

A.C. Through R-L-C Series Circuit : \rightarrow

Let a resistance of ' R '-ohm inductance of ' L ' henry and a capacitance of ' C ' farad are connected across the A.C. supply in series of applied voltage



$$e = E_m \sin \omega t \text{ -----}(1)$$

$$\begin{aligned}
\vec{e} &= \vec{V}_R + \vec{V}_L + \vec{V}_C \\
&= V_R + jV_L - jV_C \\
&= V_R + j(V_L - V_C) \\
&= I_R + j(IL - IC) \\
&= I[R + j(X_L - X_C)] \\
&= I\sqrt{R^2 + (X_L - X_C)^2} \quad \angle \pm \phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) \\
&= IZ \angle \pm \phi
\end{aligned}$$

Where $Z = \sqrt{R^2 + (X_L - X_C)^2}$ is known as the impedance of R-L-C Series Circuit.

If $X_L > X_C$, then the angle is +ve.

If $X_L < X_C$, then the angle is -ve.

Impedance is defined as the phasor sum of resistance and net reactance

$$\begin{aligned}
e &= IZ \angle \pm \phi \\
\Rightarrow I &= \frac{e}{Z \angle \pm \phi} = \frac{E_m \sin \omega t}{Z \angle \pm \phi} = I_m \sin(\omega t \pm \phi)
\end{aligned}$$

(1) If $X_L > X_C$, then P.f will be lagging.

(2) If $X_L < X_C$, then, P.f will be leading.

(3) If $X_L = X_C$, then, the circuit will be resistive one. The p.f. becomes unity and the resonance occurs.

REASONANCE

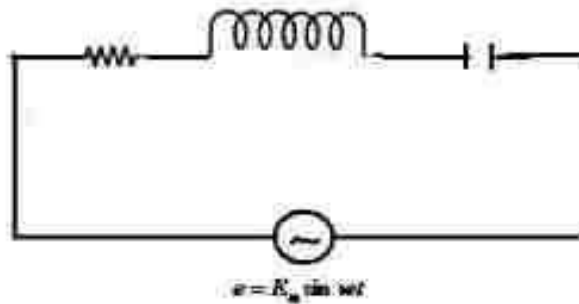
It is defined as the resonance in electrical circuit having passive or active elements represents a particular state when the current and the voltage in the circuit is maximum and minimum with respect to the magnitude of excitation at a particular frequency and the impedances being either minimum or maximum at unity power factor.

Resonance are classified into two types.

(1) Series Resonance

(2) Parallel Resonance

(1) Series Resonance :- Let a resistance of 'R' ohm, inductance of 'L' henry and capacitance of 'C' farad are connected in series across A.C. supply



$$e = E_m \sin \omega t$$

The impedance of the circuit

$$Z = R + j(X_L - X_C)$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

The condition of series resonance:

The resonance will occur when the reactive part of the line current is zero.

The p.f. becomes unity.

The net reactance will be zero.

The current becomes maximum.

At resonance net reactance is zero.

$$X_L - X_C = 0$$

$$\Rightarrow X_L = X_C$$

$$\Rightarrow \omega L = \frac{1}{\omega C}$$

$$\Rightarrow \omega^2 LC = 1$$

$$\Rightarrow \omega^2 = \frac{1}{LC}$$

$$\Rightarrow \omega = \frac{1}{\sqrt{LC}}$$

$$\Rightarrow 2\pi f_r = \frac{1}{\sqrt{LC}}$$

$$\Rightarrow f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$\text{Resonant frequency } (f_r) = \frac{1}{2\pi} \cdot \frac{1}{\sqrt{LC}}$$

Impedance at Resonance

$$Z_0 = R$$

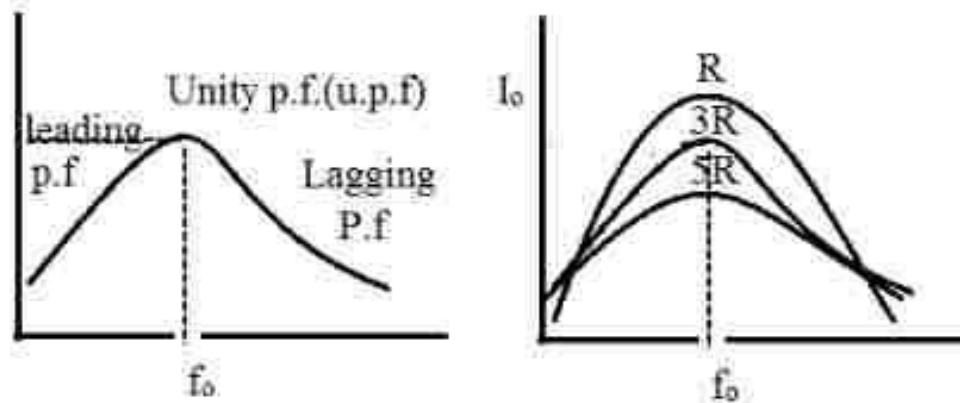
Current at Resonance

$$I_r = \frac{V}{R}$$

Power factor at resonance

$$p.f. = \frac{R}{Z_r} = \frac{R}{R} = 1 \quad [\because Z_r = R]$$

Resonance Curve :-



At low frequency the X_c is greater and the circuit behaves leading and at high frequency the X_L becomes high and the circuit behaves lagging circuit.

If the resistance will be low the curve will be stiff (peak).

- If the resistance will go on increasing the current goes on decreasing and the curve become flat.

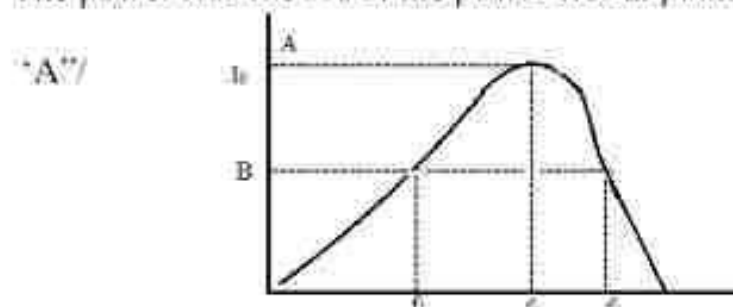
Band Width :→

At point 'A' the power loss is $I_0^2 R$.

The frequency is f_0 which is at resonance.

At point 'B' the power loss is $\frac{I_0^2 R}{2}$.

The power loss is 50% of the power loss at point



Hence the frequencies

corresponding to point 'B' is known as half power frequencies f_1 & f_2 .

f_1 = Lower half power frequency

$$f_1 = f_0 - \frac{R}{4\pi L}$$

f_2 = Upper half power frequency

$$f_2 = f_0 + \frac{R}{4\pi L}$$

Band width (B.W.) is defined as the difference between upper half power frequency and lower half power frequency.

$$B.W. = f_2 - f_1 = \frac{R}{2\pi L}$$

Selectivity : \rightarrow

Selectivity is defined as the ratio of Band width to resonant frequency

$$\text{Selectivity} = \frac{BW}{f_0} = \frac{R}{2\pi L} \quad \text{Selectivity} = \frac{R}{2\pi f_0 L}$$

Quality Factor (Q-factor) : \rightarrow

It is defined as the ratio of $2\pi \times$ Maximum energy stored to energy dissipated per cycle

$$\begin{aligned} \text{Q-factor} &= \frac{2\pi \times \frac{1}{2} LI_0^2}{I^2 RT} \\ &= \frac{\pi L (\sqrt{2} I)^2}{I^2 RT} \\ &= \frac{\pi L 2I^2}{I^2 RT} \\ &= \frac{\pi L 2I^2}{I^2 RT} \\ &= \frac{2\pi L}{RT} \end{aligned}$$

$$\text{Quality factor} = \frac{2\pi f_0 L}{R}$$

$$\left[\because \frac{1}{f} = f_0 \right]$$

Quality factor is defined as the reciprocal of power factor.

$$\text{Q factor} = \frac{1}{\cos \phi}$$

It is the reciprocal of selectivity.

$$\begin{aligned} \text{Q-factor Or Magnification factor} &= \frac{\text{Voltage across Inductor}}{\text{Voltage across resistor}} \\ &= \frac{I_0 X_L}{I_0 R} \\ &= \frac{X_L}{R} \\ &= \frac{2\pi f_0 L}{R} = \frac{W_0 L}{R} \end{aligned}$$

$$\text{Q- factor} = \frac{W_0 L}{R}$$

$$\begin{aligned} \text{Q-factor factor} &= \frac{\text{Voltage across Capacitor}}{\text{Voltage across resistor}} \\ &= \frac{I_0 X_C}{I_0 R} \end{aligned}$$

$$= \frac{X_C}{R}$$

$$= \frac{1}{2\pi f_0 C} = \frac{1}{2\pi f_0 CR}$$

$$Q\text{-factor} = \frac{1}{W_0 CR}$$

$$Q^2 = \frac{W_0 L}{R} \times \frac{1}{W_0 CR}$$

$$Q^2 = \frac{1}{R^2 C}$$

$$Q = \sqrt{\frac{1}{R^2 C}}$$

$$Q = \frac{1}{R} \sqrt{\frac{L}{C}}$$

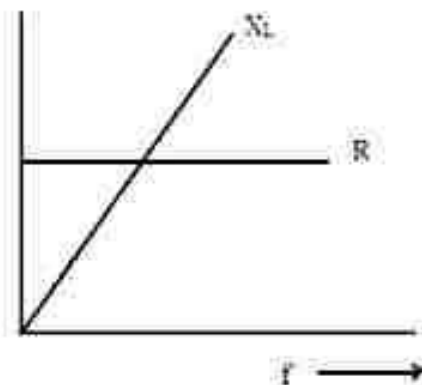
Graphical Method :→

(1) Resistance is independent of frequency It represents a straight line.

(2) Inductive Reactance $X_L = 2\pi fL$

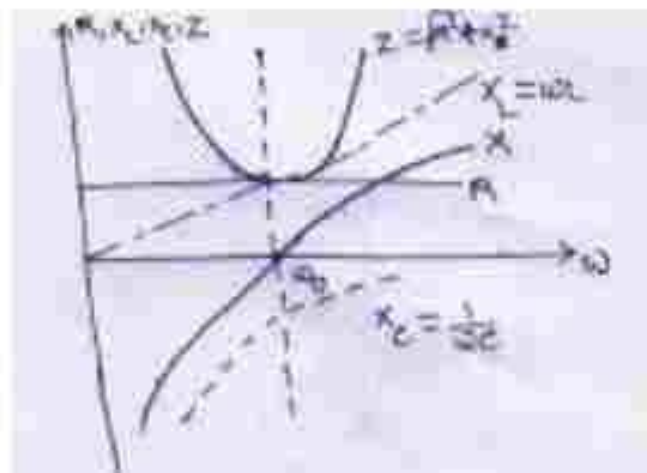
It is directly proportional to frequency. As the frequency increases, X_L increases

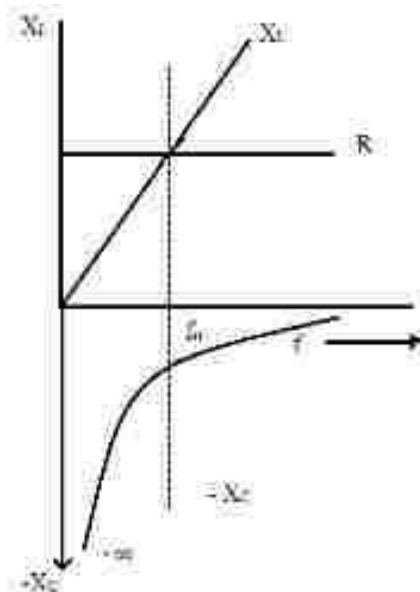
(3) Capacitive Reactance $X_C = \frac{1}{2\pi fC}$



It is inversely proportional to frequency. As the frequency increases, X_C decreases.

When frequency increases, X_L increases and X_C decreases from the higher value.



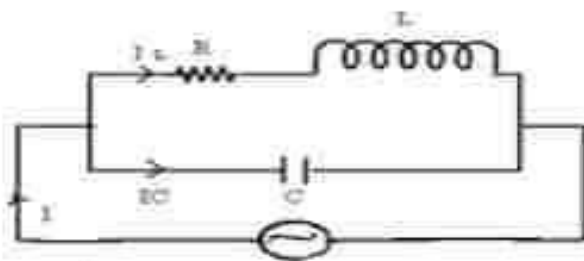


At a certain frequency, $X_L = X_C$

That particular frequency is known as Resonant frequency.

Variation of circuit parameter in series resonance:

(2) Parallel Resonance :- Resonance will occur when the reactive part of the line current is zero:



At resonance,

$$I_C - I_L \sin \phi = 0$$

$$I_C = I_L \sin \phi$$

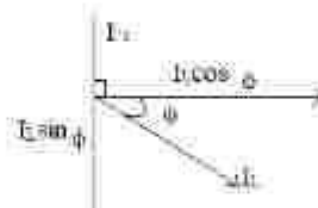
$$\Rightarrow \frac{V}{X_C} = \frac{V}{\sqrt{R^2 + X_L^2}} \sin \phi$$

$$\Rightarrow \frac{V}{X_C} = \frac{V}{\sqrt{R^2 + X_L^2}} \times \frac{X_L}{\sqrt{R^2 + X_L^2}}$$

$$\Rightarrow \frac{1}{X_C} = \frac{X_L}{R^2 + X_L^2}$$

$$\Rightarrow R^2 + X_L^2 = X_L X_C$$

$$\Rightarrow Z^2 = X_L X_C = \omega L \times \frac{1}{\omega C}$$



$$\begin{aligned}
 Z^2 &= \frac{L}{C} \\
 \Rightarrow R^2 + X_L^2 &= \frac{L}{C} \\
 \Rightarrow R^2 + (2\pi f_0 L)^2 &= \frac{L}{C} \\
 \Rightarrow R^2 + 4\pi^2 f_0^2 L^2 &= \frac{L}{C} \\
 \Rightarrow 4\pi^2 f_0^2 L^2 &= \frac{L}{C} - R^2 \\
 \Rightarrow f_0^2 &= \frac{1}{4\pi^2 f_0^2 L^2} \left(\frac{L}{C} - R^2 \right) \\
 \Rightarrow f_0 &= \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}
 \end{aligned}$$

f_0 = Resonant frequency in parallel circuit.

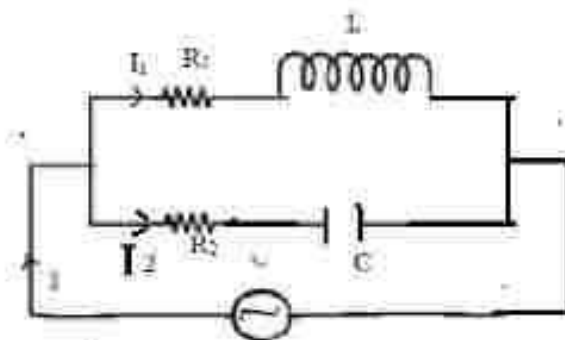
Current at Resonance = $I_L \cos \phi$

$$\begin{aligned}
 &= \frac{V}{\sqrt{R^2 + X_L^2}} \cdot \frac{R}{\sqrt{R^2 + X_L^2}} \\
 &= \frac{VR}{R^2 + X_L^2} \\
 &= \frac{VR}{Z^2} \\
 &= \frac{VR}{L/C} = \frac{V}{L/RC} \\
 &= \frac{V}{\text{Dynamic Impedance}}
 \end{aligned}$$

$L/RC \rightarrow$ Dynamic Impedance of the circuit.

or, dynamic impedances is defined as the impedance at resonance frequency in parallel circuit.

Parallel Circuit : \rightarrow



The parallel resonance condition:

When the reactive part of the line current is zero.

The net reactance is zero.

The line current will be minimum.

The power factor will be unity

Impedance $Z_1 = R_1 + jX_L$

$$Z_2 = R_2 - jX_C$$

$$\begin{aligned}\text{Admittance } Y_1 &= \frac{1}{Z_1} = \frac{1}{R_1 + jX_L} \\&= \frac{(R_1 - jX_L)}{(R_1 + jX_L)(R_1 - jX_L)} \\&= \frac{R_1 - jX_L}{R_1^2 + X_L^2} \\Y_1 &= \frac{R_1}{R_1^2 + X_L^2} - j \frac{X_L}{R_1^2 + X_L^2}\end{aligned}$$

$$\begin{aligned}\text{Admittance } Y_2 &= \frac{1}{Z_2} = \frac{1}{R_2 - jX_C} \\&= \frac{(R_2 + jX_C)}{(R_2 - jX_C)(R_2 + jX_C)} \\&= \frac{R_2 + jX_C}{R_2^2 + X_C^2} \\Y_2 &= \frac{R_2}{R_2^2 + X_C^2} + j \frac{X_C}{R_2^2 + X_C^2}\end{aligned}$$

$$\text{Total Admittance } \left(\frac{1}{Z} \right) = \frac{1}{Z_1} + \frac{1}{Z_2}$$

$$\begin{aligned}\Rightarrow Y &= Y_1 + Y_2 \\ \Rightarrow Y &= \frac{R_1}{R_1^2 + X_L^2} - j \frac{X_L}{R_1^2 + X_L^2} + \frac{R_2}{R_2^2 + X_C^2} + j \frac{X_C}{R_2^2 + X_C^2} \\ \Rightarrow Y &= \frac{R_1}{R_1^2 + X_L^2} + \frac{R_2}{R_2^2 + X_C^2} - j \left(\frac{X_L}{R_1^2 + X_L^2} - \frac{X_C}{R_2^2 + X_C^2} \right)\end{aligned}$$

At Resonance,

$$\begin{aligned}\frac{X_L}{R_1^2 + X_L^2} - \frac{X_C}{R_2^2 + X_C^2} &= 0 \\ \Rightarrow \frac{X_L}{R_1^2 + X_L^2} &= \frac{X_C}{R_2^2 + X_C^2} \\ \Rightarrow X_L (R_2^2 + X_C^2) &= X_C (R_1^2 + X_L^2) \\ \Rightarrow 2\pi fL \left(R_2^2 + \frac{1}{4\pi^2 f^2 C^2} \right) &= \frac{1}{2\pi fC} (R_1^2 + 4\pi^2 f^2 L^2) \\ \Rightarrow 2\pi fL R_2^2 + \frac{L}{2\pi fC} &= \frac{R_1^2}{2\pi fC} + \frac{2\pi fL^2}{C}\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \frac{L}{2\pi f C^2} - \frac{R_1^2}{2\pi f C} = \frac{2\pi f L^2}{C} - 2\pi f L R_2^2 \\
&\Rightarrow \frac{1}{2\pi f C} \left(\frac{L}{C} - R_1^2 \right) = 2\pi f L \left(\frac{L}{C} - R_2^2 \right) \\
&\Rightarrow 4\pi^2 f^2 LC = \frac{\frac{L}{C} - R_1^2}{\frac{L}{C} - R_2^2} = \frac{L - CR_1^2}{L - CR_2^2} \\
&\Rightarrow 4\pi^2 f^2 = \frac{1}{LC} \left(\frac{L - CR_1^2}{L - CR_2^2} \right) \\
&\Rightarrow f^2 = \frac{1}{4\pi^2 LC} \left(\frac{L - CR_1^2}{L - CR_2^2} \right) \\
&\Rightarrow f = \frac{1}{2\pi \sqrt{LC}} \sqrt{\left(\frac{L - CR_1^2}{L - CR_2^2} \right)} \\
&\Rightarrow f = \frac{1}{2\pi} \sqrt{\left(\frac{L - CR_1^2}{L^2 C - LC^2 R_2^2} \right)}
\end{aligned}$$

f is called Resonant frequency.

If $R^2 = 0$

$$\begin{aligned}
\text{Then } f &= \frac{1}{2\pi} \sqrt{\frac{L - CR_1^2}{L^2 C}} \\
&= \frac{1}{2\pi} \sqrt{\frac{L - CR_1^2}{C}} \\
&= \frac{1}{2\pi} \sqrt{\frac{L}{C} - R_1^2} \\
&= \frac{1}{2\pi} \sqrt{\frac{L}{L^2 C} - \frac{R_1^2}{L^2}}
\end{aligned}$$

$$f = \frac{1}{2\pi} \sqrt{\frac{L}{LC} - \frac{R_1^2}{L^2}}$$

If R_1 and $R_2 = 0$, then

$$f = \frac{1}{2\pi} \sqrt{\frac{L}{LC}}$$

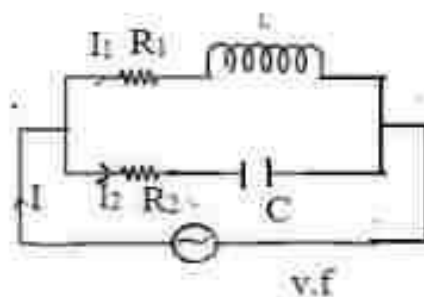
$$f = \frac{1}{2\pi} \sqrt{\frac{1}{LC}} = \frac{1}{2\pi \sqrt{LC}}$$

Comparison of Series and Parallel Resonant Circuit \rightarrow

Item	Series ckt (R-L-C)	Parallel ckt (R- L and C)
------	--------------------	---------------------------

❖ Impedance at Resonance	Minimum	Maximum
❖ Current at Resonance	Maximum = $\frac{V}{R}$	Minimum = $\frac{V}{(L/CR)}$
❖ Effective Impedance	R	$\frac{L}{CR}$
❖ P.f. at Resonance	Unity	Unity
❖ Resonant Frequency	$\frac{1}{2\pi\sqrt{LC}}$	$\frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$
❖ It Magnifies	Voltage	Current
❖ Magnification factor	$\frac{WL}{R}$	$\frac{WL}{R}$

Parallel circuit :→



$$Z_1 = R_1 + jX_L = \sqrt{R_1^2 + X_L^2} \angle \phi_1$$

$$Z_2 = R_2 - jX_C = \sqrt{R_2^2 + X_C^2} \angle -\phi_2$$

$$I_1 = \frac{V}{Z_1 \angle \phi_1} = \frac{V}{Z_1} \angle -\phi_1 = I_1 \angle -\phi_1$$

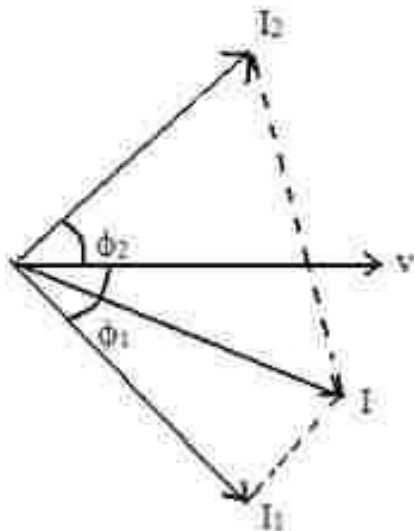
Where $\frac{V}{Z_1} = VY_1$

Here $Y_1 \rightarrow$ Admittance of the circuit

Admittance is defined as the reciprocal of impedance.

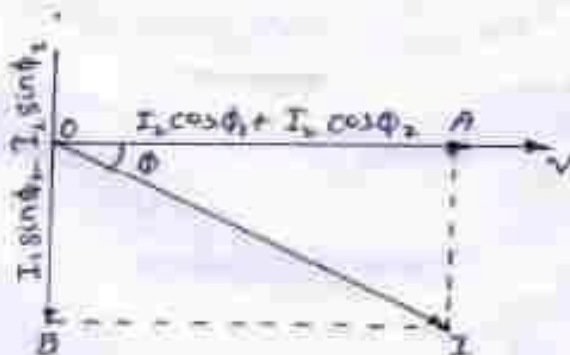
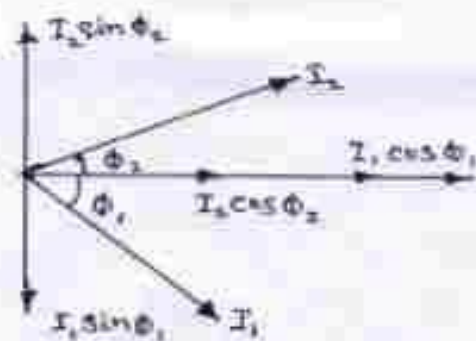
$$I_1 = VY_1 = \frac{V}{R_1 + jX_1}$$

$$I_2 = \frac{V}{Z_2 \angle -\phi_2} = \frac{V}{Z_2} \angle \phi_2 = VY_2 \angle \phi_2 = I_2 \angle \phi_2$$



$$I = \sqrt{I_1^2 + I_2^2 + 2I_1I_2 \cos(\phi_1 + \phi_2)}$$

$$I = I_1 \angle -\phi_1 + I_2 \angle \phi_2$$



The resultant current "I" is the vector sum of the branch currents I_1 & I_2 can be found by using parallelogram law of vectors or resolving I_2 into their X

– and Y- components (or active and reactive components respectively) and then by combining these components.

Sum of active components of I_1 and $I_2 = I_1 \cos \phi_1 + I_2 \cos \phi_2$

Sum of the reactive components of I_1 and $I_2 = I_2 \sin \phi_2 - I_1 \sin \phi_1$

EXP – 01 :

A 60Hz voltage of 230 V effective value is impressed on an inductance of 0.265 H

- (i) Write the time equation for the voltage and the resulting current. Let the zero axis of the voltage wave be at $t = 0$.
- (ii) Show the voltage and current on a phasor diagram.
- (iii) Find the maximum energy stored in the inductance.

Solution :-

$$V_{\max} = \sqrt{2}V = \sqrt{2} \times 230V$$

$$f = 60\text{Hz}, \quad \omega = 2\pi f = 2\pi \times 60 = 377\text{rad/s}$$

$$X_L = \omega L = 377 \times 0.265 = 100\Omega$$

- (i) The time equation for voltage is $V(t) = 230\sqrt{2} \sin 377t$.

$$I_{\max} = V_{\max} / X_L = 230\sqrt{2} / 100 = 2.3\sqrt{2}$$

$$\phi = 90^\circ (\text{lag}).$$

\therefore Current equation is:

$$I(t) = 2.3\sqrt{2} \sin(377t - \pi/2)$$

$$\text{or } I = 2.3\sqrt{2} \cos 377t$$

- (ii) Iti

- (iii) or $E_{\max} = \frac{1}{2}LI^2_{\max} = \frac{1}{2} \times 0.265 \times (2.3\sqrt{2})^2 = 1.4J$

Example -02 :

The potential difference measured across a coil is 4.5 v, when it carries a direct current of 9 A. The same coil when carries an alternating current of 9A at 25 Hz, the potential difference is 24 v. Find the power and the power factor when it is supplied by 50 v, 50 Hz supply.

Solution :

Let R be the d.c. resistance and L be inductance of the coil.

$$R = V / I = 4.5 / 9 = 0.5\Omega$$

With a.c. current of 25Hz, $z = V/I$.

$$\frac{24}{9} = 2.66\Omega$$

$$x_L = \sqrt{Z^2 - R^2} = \sqrt{2.66^2 - 0.5^2}$$

$$= 2.62\Omega$$

$$x_L = 2\pi \times 25 \times L$$

$$x_L = 0.0167\Omega$$

At 50Hz

$$x_L = 2.62 \times 2 = 5.24\Omega$$

$$Z = \sqrt{0.5^2 + 5.24^2}$$

$$= 5.06\Omega$$

$$I = 50/5.26 = 9.5 \text{ A}$$

$$P = I^2/R = 9.5^2 \times 0.5 = 45 \text{ watt.}$$

Example – 03 :

A 50- μf capacitor is connected across a 230-v, 50 – Hz supply. Calculate

- The reactance offered by the capacitor.
- The maximum current and
- The r.m.s value of the current drawn by the capacitor.

Solution :

$$(a) \quad x_C = \frac{1}{\omega C} = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 50 \times 10^{-6}} = 63.6\Omega$$

$$(c) \quad \text{Since 230 v represents the r.m.s value}$$

$$\therefore I_{\text{rms}} = 230 / x_C = 230 / 63.6 = 3.62 \text{ A}$$

$$(b) \quad I_m = I_{\text{rms}} \times \sqrt{2} = 3.62 \times \sqrt{2} = 5.11 \text{ A}$$

Example – 04 :

In a particular R – L series circuit a voltage of 10v at 50 Hz produces a current of 700 mA. What are the values of R and L in the circuit ?

Solution :

$$(i) \quad Z = \sqrt{R^2 + (2\pi \times 50L)^2}$$

$$= \sqrt{R^2 + 98696L^2}$$

$$V = Iz$$

$$10 = 700 \times 10^{-3} \sqrt{R^2 + 98696L^2}$$

$$\sqrt{R^2 + 98696L^2} = 10 / 700 \times 10^{-3} = 100 / 7$$

$$R^2 + 98696L^2 = 10000 / 49 \text{----- (1)}$$

$$(ii) \quad \text{In the second case } Z = \sqrt{R^2 + (2\pi \times 75L)^2}$$

$$\therefore 10 = 500 \times 10^{-3} \sqrt{R^2 + 222066L^2} = 20$$

$$\sqrt{R^2 + 222066L^2} = 20$$

CHAPTER-07

TRANSIENTS

Whenever a network containing energy storage elements such as inductor or capacitor is switched from one condition to another, either by change in applied source or change in network elements, the response current and voltage change from one state to the other state. The time taken to change from an initial steady state to the final steady state is known as the *transient period*. This response is known as *transient response* or *transients*. The response of the network after it attains a final steady value is independent of time and is called the steady-state response. The complete response of the network is determined with the help of a differential equation.

STEADY STATE AND TRANSIENT RESPONSE

In a network containing energy storage elements, with change in excitation, the currents and voltages in the circuit change from one state to other state. The behaviour of the voltage or current when it is changed from one state to another is called the transient state. The time taken for the circuit to change from one steady state to another steady state is called the transient time. The application of KVL and KCL to circuits containing energy storage elements results in differential, rather than algebraic equations. When we consider a circuit containing storage elements which are independent of the sources, the response depends upon the nature of the circuit and is called natural response. Storage elements deliver their energy to the resistances. Hence, the response changes, gets saturated after some time, and is referred to as the transient response. When we consider a source acting on a circuit, the response depends on the nature of the source or sources. This response is called forced response. In other words, the complete response of a circuit consists of two parts; the forced response and the transient response. When we consider a differential equation, the complete solution consists of two parts; the complementary function and the particular solution. The complementary function dies out after short interval, and is referred to as the transient response or source free response. The particular solution is the steady state response, or the forced response. The first step in finding the complete solution of a circuit is to form a differential equation for the circuit. By obtaining the differential equation, several methods can be used to find out the complete solution.

DC RESPONSE OF AN R-L CIRCUIT

Consider a circuit consisting of a resistance and inductance as shown in figure. The inductor in the circuit is initially uncharged and is in series with the resistor. When the switch S is closed, we can find the complete solution for the current. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

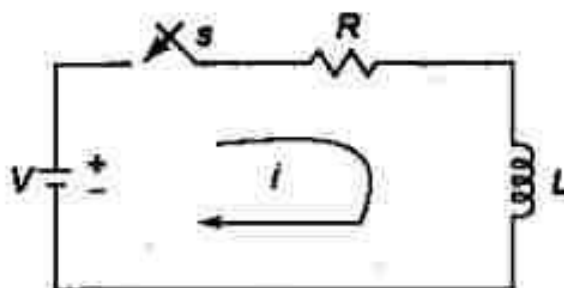


Figure 1.1

$$V = Ri + L \frac{di}{dt} \quad \dots\dots\dots 1.1$$

$$\text{Or } \frac{di}{dt} + \frac{R}{L} i = \frac{V}{L} \quad \dots\dots\dots 1.2$$

In the above equation, the current i is the solution to be found and V is the applied constant voltage. The voltage V is applied to the circuit only when the switch S is closed. The above equation is a linear differential equation of first order comparing it with a non-homogenous differential equation

$$\frac{dx}{dt} + P \cdot x = K \quad \dots\dots\dots 1.3$$

whose solution is

$$x = e^{-Pt} \left[\int K e^{+Pt} dt + c \right] e^{-Pt} \quad \dots\dots\dots 1.4$$

Where c is an arbitrary constant. In a similar way, we can write the current equation as

$$i = c e^{-\left(\frac{R}{L}\right)t} + e^{-\left(\frac{R}{L}\right)t} \int \frac{V}{L} e^{\left(\frac{R}{L}\right)t} dt$$

$$\text{Hence, } i = c e^{-\left(\frac{R}{L}\right)t} + \frac{V}{R} \quad \dots\dots\dots 1.5$$

To determine the value of c in equation 1.5, we use the initial conditions. In the circuit shown in Fig. 1.1, the switch s is closed at $t=0$. At $t=0^-$, i.e. just before closing the switch s , the current in the inductor is zero. Since the inductor does not allow sudden changes in currents, at $t=0^+$ just after the switch is closed, the current remains zero.

Thus at $t = 0$, $i = 0$

Substituting the above condition in equation 1.5, we have

$$0 = c + \frac{V}{R}$$

Substituting the value of c in equation 1.5, we get

$$i = \frac{V}{R} - \frac{V}{R} e^{-\frac{Rt}{L}}$$

$$i = \frac{V}{R} \left(1 - e^{-\frac{Rt}{L}} \right)$$

$$i = I_0 \left(1 - e^{-\frac{Rt}{L}} \right) \text{ (where } I_0 = \frac{V}{R} \text{)}$$

$$i = I_0 \left(1 - e^{-\frac{t}{\tau}} \right) \text{ (where } \tau = \text{Time constant} = \frac{L}{R} \text{)} \quad \dots\dots\dots 1.6$$

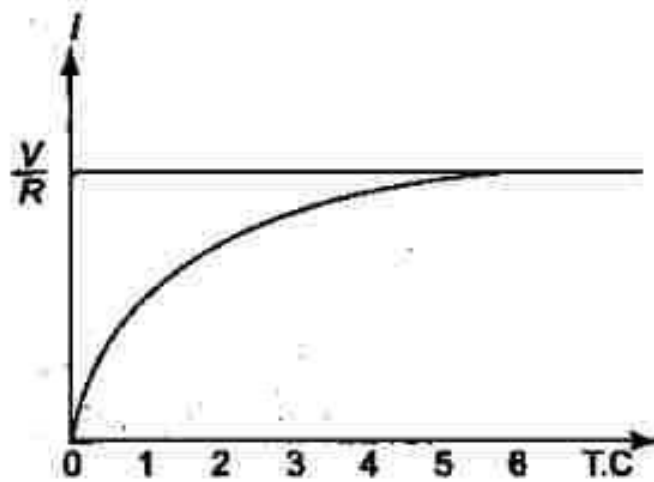


Figure 1.2

Equation d consists of two parts, the steady state part ($I_{\infty} = V/R$) and the transient part $I_{\infty} e^{-\frac{Rt}{L}}$.

When switch S is closed, the response reaches a steady state value after a time interval as shown in figure 1.2.

Here the transition period is defined as the time taken for the current to reach its final or steady state value from its initial value. In the transient part of the solution, the quantity L/R is important in describing the curve since L/R is the time period required for the current to reach its initial value of zero to the final value $I_{\infty} = V/R$. The time constant of a function $I_{\infty} e^{-\frac{Rt}{L}}$ is the time at which the exponent of e is unity, where e is the base of the natural logarithms. The term L/R is called the time constant and is denoted by τ .

$$\text{So, } \tau = \frac{L}{R} \text{ sec}$$

Hence, the transient part of the solution is

$$i = -\frac{V}{R} e^{-\frac{Rt}{L}} = -\frac{V}{R} e^{-\frac{t}{\tau}}$$

At one Time constant, the transient term reaches 36.8 percent of its initial value.

$$i(\tau) = -\frac{V}{R} e^{-1} = -\frac{V}{R} e^{-1} = -0.368 \frac{V}{R}$$

Similarly,

$$i(2\tau) = -\frac{V}{R} e^{-2} = -0.135 \frac{V}{R}$$

$$i(3\tau) = -\frac{V}{R} e^{-3} = -0.0498 \frac{V}{R}$$

$$i(5\tau) = -\frac{V}{R} e^{-5} = -0.0067 \frac{V}{R}$$

After 5 TC the transient part reaches more than 99 percent of its final value.

In figure A we can find out the voltages and powers across each element by using the current.

Voltage across the resistor is

$$v_R = Ri = R \times \frac{V}{R} \left(1 - e^{-\frac{Rt}{L}}\right)$$

Hence, $v_R = V \left(1 - e^{-\frac{Rt}{L}}\right)$

Similarly, the voltage across the inductance is

$$v_L = L \frac{di}{dt} = L \times \frac{V}{R} \times \frac{R}{L} e^{-\frac{Rt}{L}} = V e^{-\frac{Rt}{L}}$$

The responses are shown in Figure 1.3.

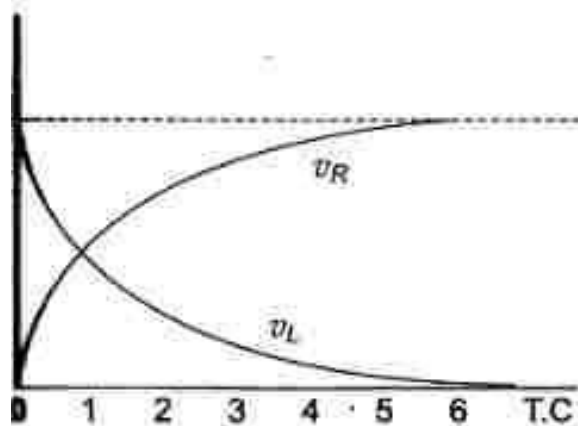


Figure 1.3

Power in the resistor is

$$\begin{aligned} P_R = v_R i &= V \left(1 - e^{-\frac{Rt}{L}}\right) \left(1 - e^{-\frac{Rt}{L}}\right) \times \frac{V}{R} \\ &= \frac{V^2}{R} \left(1 - 2e^{-\frac{Rt}{L}} + e^{-\frac{2Rt}{L}}\right) \end{aligned}$$

Power in the inductor is

$$\begin{aligned} P_L = v_L i &= V e^{-\frac{Rt}{L}} \times \frac{V}{R} \left(1 - e^{-\frac{Rt}{L}}\right) \\ &= \frac{V^2}{R} \left(e^{-\frac{Rt}{L}} - e^{-\frac{2Rt}{L}}\right) \end{aligned}$$

The responses are shown in figure 1.4.

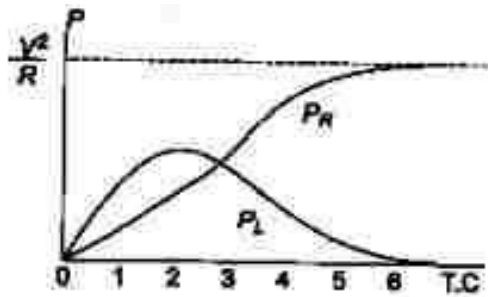


Figure 1.4

Problem : 1.1

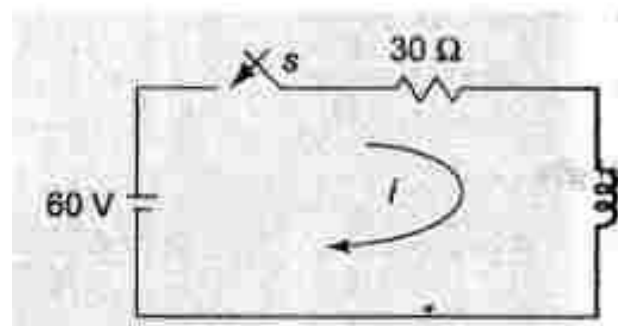


Figure 1.5

A series R-L circuit with $R = 30\Omega$ and $L = 15\text{ H}$ has a constant voltage $V = 50\text{ V}$ applied at $t=0$ as shown in Fig. 1.5 . determine the current i , the voltage across resistor and across inductor.

Solution :

By applying Kirchhoff's voltage Law, we get

$$15 \frac{di}{dt} + 30i = 60$$

$$\Rightarrow \frac{di}{dt} + 2i = 4$$

The general solution for a linear differential equation is

$$i = ce^{-Pt} + e^{-Pt} \int K e^{Pt} dt$$

where $P=2, K=4$

putting the values

$$i = ce^{-2t} + e^{-2t} \int 4e^{2t} dt$$

$$\Rightarrow i = ce^{-2t} + 2$$

At $t=0$, the switch s is closed.

Since the inductor never allows sudden change in currents. At $t=0^+$ the current in the circuit is zero. Therefore at $t=0^+$, $i=0$

$$\Rightarrow 0=c+2$$

$$\Rightarrow c=-2$$

Substituting the value of c in the current equation, we have

$$i=2(1-e^{-2t}) \text{ A}$$

$$\text{voltage across resistor } (V_R) = iR = 2(1-e^{-2t}) \times 30 = 60(1-e^{-2t}) \text{ V}$$

$$\text{voltage across inductor } (V_L) = L \frac{di}{dt} = 15 \times \frac{d}{dt} 2(1-e^{-2t}) = 30 \times 2e^{-2t} \text{ V} = 60e^{-2t}$$

DC RESPONSE OF AN R-C CIRCUIT

Consider a circuit consisting of a resistance and capacitance as shown in figure. The capacitor in the circuit is initially uncharged and is in series with the resistor. When the switch S is closed at $t=0$, we can find the complete solution for the current. Application of kirchoff's voltage law to the circuit results in the following differential equation.

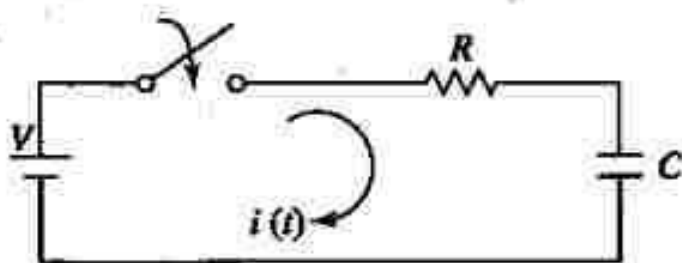


Figure 1.6

$$V = Ri + \frac{1}{C} \int i \, dt \quad \dots\dots\dots 1.7$$

By differentiating the above equation, we get

$$0 = R \frac{di}{dt} + \frac{1}{C} i \quad \dots\dots\dots 1.8$$

Or

$$\frac{di}{dt} + \frac{1}{RC} i = 0 \quad \dots\dots\dots 1.9$$

Equation c is a linear differential equation with only the complementary function. The particular solution for the above equation is zero. The solution for this type of differential equation is

$$i = c e^{-\left(\frac{t}{RC}\right)} \dots\dots\dots 1.10$$

To determine the value of c in equation c, we use the initial conditions. In the circuit shown in Fig. the switch s is closed at t=0. Since the capacitor does not allow sudden changes in voltage, it will act as a short circuit at t=0+ just after the switch is closed.

So the current in the circuit at t = 0+ is $\frac{V}{R}$

Thus at t = 0, the current $i = \frac{V}{R}$

Substituting the above condition in equation c, we have

$$\frac{V}{R} = c$$

Substituting the value of c in equation c, we get

$$i = \frac{V}{R} e^{-\left(\frac{t}{RC}\right)} \dots\dots\dots 1.11$$

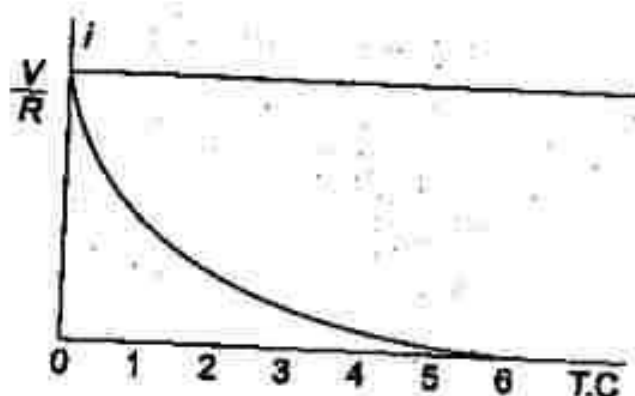


Figure 1.7

When switch S is closed, the response decays as shown in figure.

The term RC is called the time constant and is denoted by τ .

So, $\tau = RC$ sec

After 5 TC the curve reaches 99 percent of its final value.

In figure A we can find out the voltage across each element by using the current equation.

Voltage across the resistor is

$$v_R = Ri = R \times \frac{V}{R} e^{-\frac{t}{RC}}$$

Hence, $v_R = V e^{-\frac{t}{RC}}$

Similarly, voltage across the capacitor is

$$v_C = \frac{1}{C} \int i \, dt$$

$$= \frac{1}{C} \int \frac{V}{R} e^{-\frac{t}{RC}} \, dt$$

$$= - \left(\frac{V}{RC} \times RC e^{-\frac{t}{RC}} \right) + c$$

$$= -V e^{-\frac{t}{RC}} + c$$

At $t=0$, voltage across capacitor is zero

So, $c = V$

And

$$V_C = V(1 - e^{-\frac{t}{RC}})$$

The responses are shown in Figure 1.8.

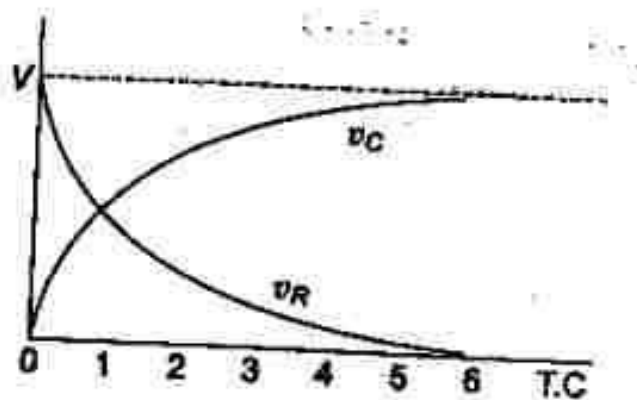


Figure 1.8

Power in the resistor is

$$P_R = v_R i = V e^{-\frac{t}{RC}} \times \frac{V}{R} e^{-\frac{t}{RC}}$$

$$= \frac{V^2}{R} e^{-\frac{2t}{RC}}$$

Power in the capacitor is

$$P_C = v_C i = V(1 - e^{-\frac{t}{RC}}) \frac{V}{R} e^{-\frac{t}{RC}}$$

$$= \frac{1}{R} \left(e^{-\frac{t}{RC}} - e^{-\frac{t}{RC}} \right)$$

The responses are shown in figure 1.9.

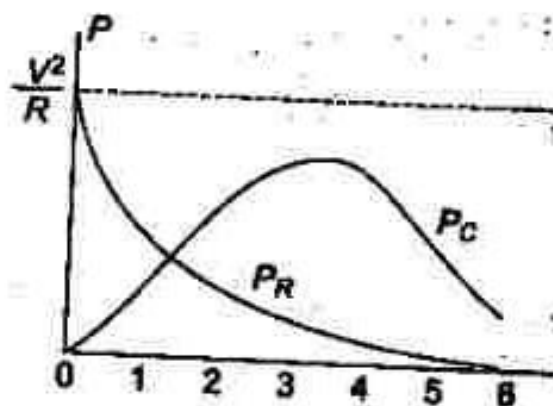


Figure 1.9

Problem : 1.2

A series R-C circuit with $R = 10\Omega$ and $C = 0.1\text{ F}$ has a constant voltage $V = 20\text{ V}$ applied at $t=0$ as shown in Fig. determine the current i , the voltage across resistor and across capacitor.

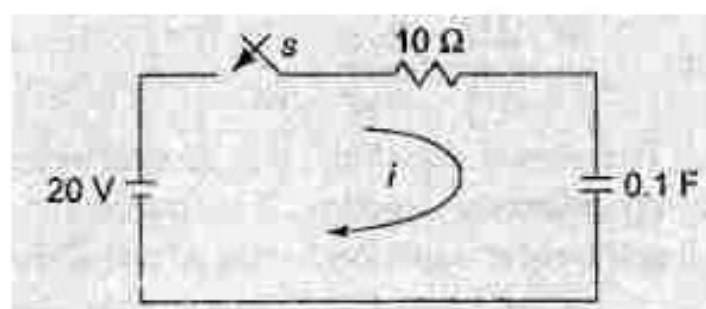


Figure 1.10

Solution :

By applying Kirchoff's voltage Law, we get

$$10i + \frac{1}{0.1} \int i dt = 20$$

Differentiating w.r.t. t we get

$$10 \frac{di}{dt} + \frac{1}{0.1} i = 0$$

$$\Rightarrow \frac{di}{dt} + i = 0$$

The solution for above equation is

$$i = Ce^{-t/\tau}$$

At $t=0$, the switch s is closed.

Since the capacitor never allows sudden change in voltages. At $t=0^+$ the current in the circuit is $i = V/R = 20/10 = 2 \text{ A}$

Therefore at $t=0$, $i = 2 \text{ A}$

\Rightarrow the current equation is $i = 2e^{-t/\tau}$

voltage across resistor (V_R) $= iR = 2e^{-t/\tau} \times 10 = 20e^{-t/\tau} \text{ V}$

voltage across capacitor (V_C) $= V(1 - e^{-t/\tau}) = 20(1 - e^{-t/\tau}) \text{ V}$

DC RESPONSE OF AN R-L-C CIRCUIT

Consider a circuit consisting of a resistance, inductance and capacitance as shown in figure. The capacitor and inductor in the circuit is initially uncharged and are in series with the resistor. When the switch S is closed at $t=0$, we can find the complete solution for the current. Application of kirchoff's voltage law to the circuit results in the following differential equation.

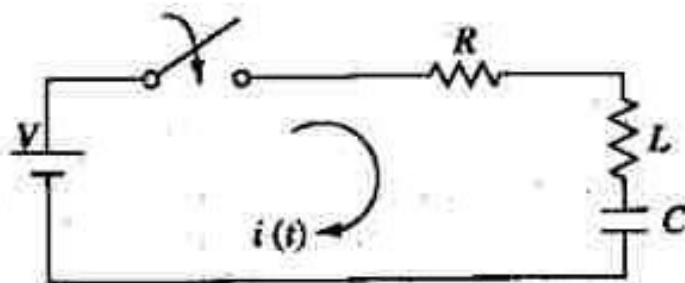


Figure 1.11

$$V = Ri + L \frac{di}{dt} + \frac{1}{C} \int i dt \quad \dots\dots\dots 1.12$$

By differentiating the above equation, we get

$$0 = R \frac{di}{dt} + L \frac{d^2 i}{dt^2} + \frac{1}{C} i = \quad \dots\dots\dots 1.13$$

Or,

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = 0 \quad \dots\dots\dots 1.14$$

The above equation is a second order linear differential equation with only the complementary function. The particular solution for the above equation is zero. The characteristics equation for this type of differential equation is

$$D^2 + \frac{R}{L}D + \frac{1}{LC} = 0 \quad \text{1.15}$$

The roots of equation 1.15 are

$$D_1, D_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\text{By assuming } K_1 = -\frac{R}{2L} \text{ and } K_2 = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$D_1 = K_1 - K_2 \text{ and } D_2 = K_1 + K_2$$

Here K_2 may be positive, negative or zero.

$$\text{Case I: } K_2 \text{ is Positive } \left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$$

Then, the roots are Real and Unequal and give an over damped Response as shown in figure 1.12.

The solution for the above equation is: $i = C_1 e^{(K_1 - K_2)t} + C_2 e^{(K_1 + K_2)t}$

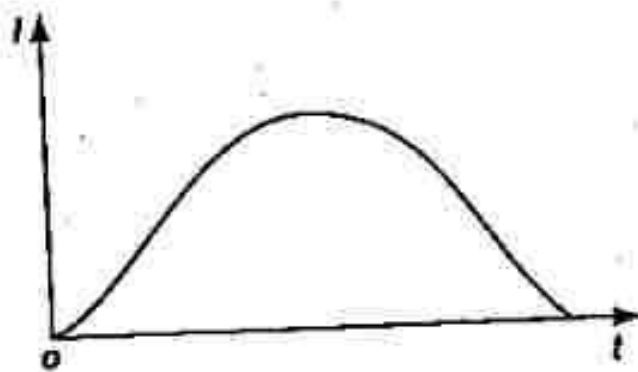


Figure 1.12

$$\text{Case II: } K_2 \text{ is Negative } \left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$$

Then, the roots are Complex Conjugate, and give an under-damped Response as shown in figure 1.13.

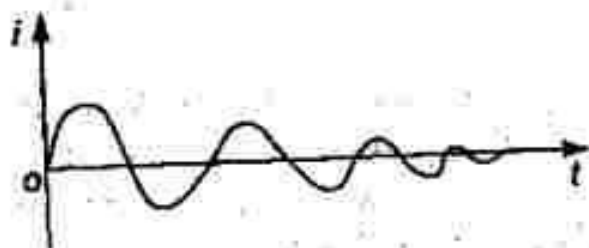


Figure 1.13

The solution for the above equation is: $i = e^{-K_1 t} (C_1 \cos K_2 t + C_2 \sin K_2 t)$

Case III: K_2 is Zero $\left(\frac{R}{L}\right)^2 = \frac{1}{LC}$

Then, the roots are Equal and give an Critically-damped Response as shown in figure 1.14.

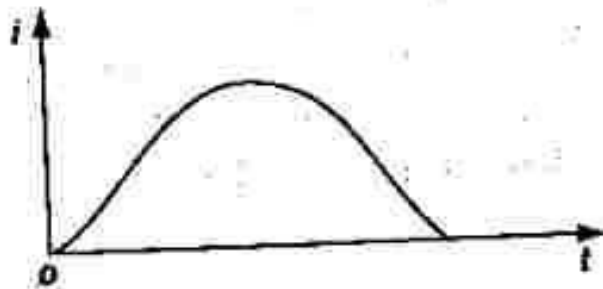


Figure 1.14

The solution for the above equation is: $i = e^{-K_1 t} (C_1 + C_2 t)$

Problem : 1.3

A series R-L-C circuit with $R = 20\Omega$, $L = 0.05\text{H}$ and $C = 20\mu\text{F}$ has a constant voltage $V = 100\text{V}$ applied at $t=0$ as shown in Fig. determine the transient current i .

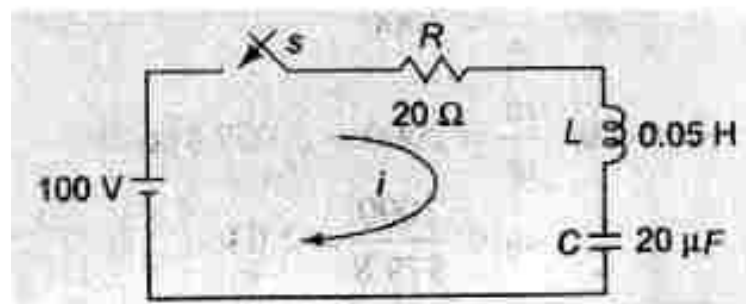


Figure 1.15

Solution :

By applying Kirchoff's voltage Law, we get

$$100 = 30i + 0.05 \frac{di}{dt} + \frac{1}{20 \times 10^{-6}} \int i dt$$

Differentiating w.r.t. t we get

$$0.05 \frac{d^2 i}{dt^2} + 20 \frac{di}{dt} + \frac{1}{20 \times 10^{-6}} i = 0$$

$$\Rightarrow d^2 i / dt^2 + 400 \frac{di}{dt} + 10^6 i = 0$$

$$\Rightarrow (D^2 + 400D + 10^6)i = 0$$

The roots of equation are

$$\begin{aligned} D_1, D_2 &= -\frac{400}{2} \pm \sqrt{\left(\frac{400}{2}\right)^2 - 10^6} \\ &= -200 \pm \sqrt{(200)^2 - 10^6} \end{aligned}$$

$$D_1 = -200 + j979.8$$

$$D_2 = -200 - j979.8$$

Therefore the current

$$i = e^{-200t} [C_1 \cos 979.8 t + C_2 \sin 979.8 t]$$

$$i = e^{-200t} [C_1 \cos 979.8 t + C_2 \sin 979.8 t] \text{ A}$$

At $t=0$, the switch s is closed.

Since the inductor never allows sudden change in currents. At $t=0^+$ the current in the circuit is zero. Therefore at $t=0^+$, $i=0$

$$\Rightarrow i=0 = (1) [C_1 \cos 0 + C_2 \sin 0]$$

$$\Rightarrow C_1 = 0 \text{ and } i = e^{-200t} [C_2 \sin 979.8 t] \text{ A}$$

Differentiating w.r.t. t we get

$$\frac{di}{dt} = C_2 [e^{-200t} 979.8 \cos 979.8 t + e^{-200t} (-200) \sin 979.8 t]$$

At $t=0$, the voltage across the inductor is 100 V

$$\Rightarrow L \frac{di}{dt} = 100 \text{ or } \frac{di}{dt} = 2000$$

$$\text{At } t=0, \frac{di}{dt} = 2000 = C_2 979.8 \cos 0$$

$$\Rightarrow C_2 = \frac{2000}{979.8} = 2.04$$

The current equation is

$$i = e^{-200t} (2.04 \sin 979.6t) \text{ A}$$

ANALYSIS OF CIRCUITS USING LAPLACE TRANSFORM TECHNIQUE

The Laplace transform is a powerful Analytical Technique that is widely used to study the behavior of Linear Lumped parameter circuits. Laplace Transform converts a time domain function $f(t)$ to a frequency domain function $F(s)$ and also Inverse Laplace transformation converts the frequency domain function $F(s)$ back to a time domain function $f(t)$.

$$L\{f(t)\} = F(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt \quad \text{LT 1}$$

$$L^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} F(s) e^{st} ds \quad \text{LT 2}$$

DC RESPONSE OF AN R-L CIRCUIT (LT Method)

Let us determine the solution i of the first order differential equation given by equation A which is for the DC response of a R-L Circuit under the zero initial condition i.e. current is zero, $i=0$ at $t=0^-$ and hence $i=0$ at $t=0^+$ in the circuit in figure A by the property of Inductance not allowing the current to change as switch is closed at $t=0$.

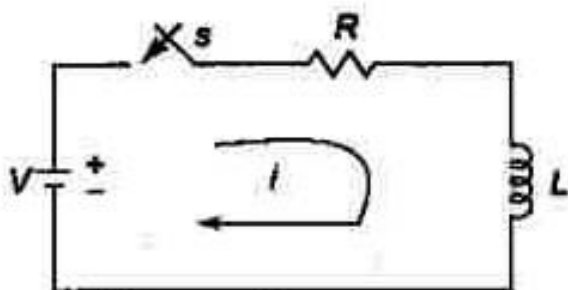


Figure LT 1.1

$$V = Ri + L \frac{di}{dt} \quad \text{LT 1.1}$$

Taking the Laplace Transform of both sides we get,

$$\frac{V}{s} = R I(s) + L [s I(s) - i(0)] \quad \text{LT 1.2}$$

$$\Rightarrow \frac{V}{s} = R I(s) + L [s I(s)] \quad (i(0) = 0 \text{ : zero initial current})$$

$$\Rightarrow \frac{V}{s} = I(s) [R + Ls]$$

$$\Rightarrow I(s) = \frac{V}{s(R + Ls)} \quad \text{LT 1.3}$$

Taking the Laplace Inverse Transform of both sides we get,

$$\Rightarrow \mathcal{L}^{-1}\{I(s)\} = i(t) = \mathcal{L}^{-1}\left\{\frac{V}{s(R+Ls)}\right\}$$

$$i(t) = \mathcal{L}^{-1}\left\{\frac{V/L}{s(\frac{R}{L}+s)}\right\} \quad (\text{Dividing the numerator and denominator by } L)$$

putting $\alpha = R/L$ we get

$$i(t) = \mathcal{L}^{-1}\left\{\frac{V/L}{s(s+\alpha)}\right\} = \mathcal{L}^{-1}\left\{\frac{V}{L}\left(\frac{1}{s} - \frac{1}{(s+\alpha)}\right)\frac{1}{\alpha}\right\}$$

$$i(t) = \mathcal{L}^{-1}\left\{\frac{V}{L}\left(\frac{1}{s} - \frac{1}{(s+R/L)}\right)\frac{1}{R}\right\} \quad (\text{again putting back the value of } \alpha)$$

$$i(t) = \mathcal{L}^{-1}\left\{\frac{V}{R}\left(\frac{1}{s} - \frac{1}{(s+R/L)}\right)\right\} = \frac{V}{R}\left(1 - e^{-\frac{Rt}{L}}\right) = I_0\left(1 - e^{-\frac{t}{\tau}}\right) \quad (\text{where } I_0 = \frac{V}{R})$$

$$i(t) = I_0\left(1 - e^{-\frac{t}{\tau}}\right) \quad (\text{where } \tau = \text{Time constant} = \frac{L}{R}) \quad \dots\dots\dots \text{LT 1.4}$$

It can be observed that solution for $i(t)$ as obtained by Laplace Transform technique is same as that obtained by standard differential method.

DC RESPONSE OF AN R-C CIRCUIT(L.T.Method)

Similarly ,

Let us determine the solution i of the first order differential equation given by equation A which is for the DC response of a R-C Circuit under the zero initial condition i.e. voltage across capacitor is zero, $V_c = 0$ at $t=0^-$ and hence $V_r = 0$ at $t=0^+$ in the circuit in figure A by the property of capacitance not allowing the voltage across it to change as switch is closed at $t=0$.

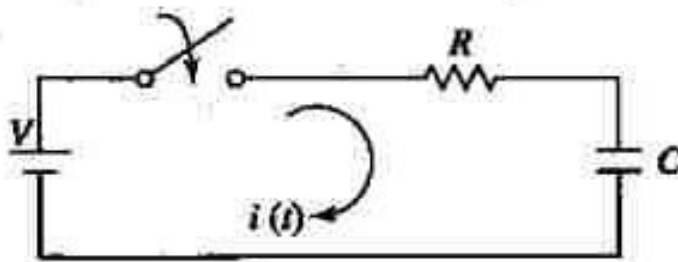


Figure LT 1.2

$$V = Ri + \frac{1}{C} \int i \, dt \quad \dots\dots\dots \text{LT 1.5}$$

Taking the Laplace Transform of both sides we get,

$$\frac{V}{s} = R I(s) + \frac{1}{C} \left[\frac{Q_0}{s} + I(0) \right] \quad \dots\dots\dots \text{LT 1.6}$$

$$\Rightarrow \frac{V}{s} = R I(s) + \frac{1}{C} \left[\frac{Q_0}{s} \right] \quad (I(0)=0 : \text{zero initial charge})$$

$$\Rightarrow \frac{V}{s} = I(s) \left[R + \frac{1}{Cs} \right] = I(s) \left[\frac{RCs+1}{Cs} \right]$$

$$\Rightarrow I(s) = \frac{V}{s} \left[\frac{Cs}{(RCs+1)} \right] = \frac{VC}{(RCs+1)} \dots\dots\dots \text{LT 1.7}$$

Taking the Laplace Inverse Transform of both sides we get,

$$\Rightarrow \mathcal{L}^{-1}\{I(s)\} = i(t) = \mathcal{L}^{-1}\left\{\frac{VC}{(RCs+1)}\right\}$$

$$i(t) = \mathcal{L}^{-1}\left\{\frac{\frac{VC}{RC}}{s+\frac{1}{RC}}\right\} \quad (\text{Dividing the numerator and denominator by } RC)$$

putting $\alpha = \frac{1}{RC}$ we get

$$i(t) = \mathcal{L}^{-1}\left\{\frac{\frac{V/R}{s+\alpha}}{s+\alpha}\right\} = \frac{V}{R} e^{-\alpha t}$$

$$i(t) = \frac{V}{R} e^{-\frac{t}{RC}} \quad (\text{putting back the value of } \alpha)$$

$$i(t) = I_0 e^{-\frac{t}{\tau}} \quad (\text{where } I_0 = \frac{V}{R}) \dots\dots\dots \text{LT 1.8}$$

$$i(t) = I_0 e^{-\frac{t}{\tau}} \quad (\text{where } \tau = \text{Time constant} = RC)$$

It can be observed that solution for $i(t)$ as obtained by Laplace Transform technique in a is same as that obtained by standard differential method in d.

DC RESPONSE OF AN R-L-C CIRCUIT (L.T. Method)

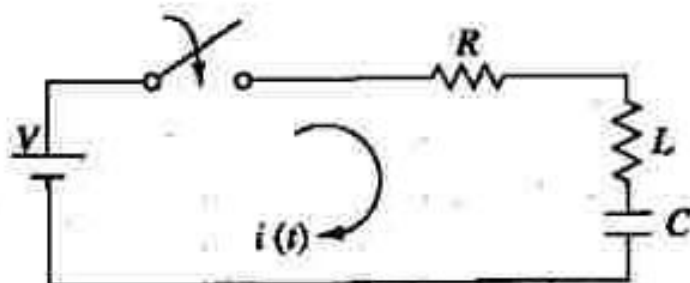


Figure LT 1.3

Similarly ,

Let us determine the solution i of the first order differential equation given by equation A which is for the DC response of a R-L-C Circuit under the zero initial condition i.e. the switch s is closed at $t=0$. at $t=0^-$ i.e. just before closing the switch s , the current in the inductor is zero. Since the inductor does not allow sudden changes in currents, at $t=0^+$ just after the switch is closed, the current remains zero. also the voltage across capacitor is zero i.e. $V_C = 0$ at $t=0^-$ and hence $V_C = 0$ at $t=0^+$ in the circuit in figure by the property of capacitance not allowing the voltage across it V_C to suddenly change as switch is closed at $t=0$.

$$V = Ri + L \frac{di}{dt} + \frac{1}{C} \int i dt \dots\dots\dots \text{LT 1.9}$$

Taking the Laplace Transform of both sides we get,

$$\frac{V}{s} = R I(s) + L [s I(s) - I(0)] + \frac{1}{C} \left[\frac{V(s)}{s} + I(0) \right] \quad \text{.....LT 1.10}$$

$$\Rightarrow \frac{V}{s} = R I(s) + L [s I(s)] + \frac{1}{C} \left[\frac{V(s)}{s} \right] \quad (I(0) = 0; \text{zero initial current} \ \& \ I(0) = 0; \text{zero initial charge})$$

$$\Rightarrow \frac{V}{s} = I(s) \left[R + Ls + \frac{1}{Cs} \right] = I(s) \left[\frac{LCs^2 + RCs + 1}{Cs} \right]$$

$$\Rightarrow I(s) = \frac{V}{s} \left[\frac{Cs}{LCs^2 + RCs + 1} \right] = \frac{VC}{(LCs^2 + RCs + 1)} \quad \text{.....LT 1.11}$$

Taking the Laplace Inverse Transform of both sides we get,

$$\Rightarrow \mathcal{L}^{-1}\{I(s)\} = i(t) = \mathcal{L}^{-1}\left\{\frac{VC}{(LCs^2 + RCs + 1)}\right\}$$

$$i(t) = \mathcal{L}^{-1}\left\{\frac{\frac{VC}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\right\} \quad (\text{Dividing the numerator and denominator by LC})$$

$$i(t) = \mathcal{L}^{-1}\left\{\frac{\frac{V}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\right\}$$

$$\text{putting } \alpha = \frac{R}{2L} \quad \text{and} \quad \omega = \sqrt{\frac{1}{LC}} \quad \text{we get}$$

$$i(t) = \mathcal{L}^{-1}\left\{\frac{\frac{V}{L}}{s^2 + 2\alpha s + \omega^2}\right\}$$

The denominator polynomial becomes = $[s^2 + 2\alpha s + \omega^2]$

$$\text{where, } s_1, s_2 = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2}}{2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2} = -\alpha \pm \beta$$

$$\text{where, } \alpha = \frac{R}{2L}; \quad \omega = \sqrt{\frac{1}{LC}} \quad \text{and} \quad \beta = \sqrt{\alpha^2 - \omega^2}$$

By partial fraction expansion, of $I(s)$,

$$I(s) = \frac{A}{s-s_1} + \frac{B}{s-s_2}$$

$$A = (s-s_1) I(s) \big|_{s=s_1}$$

$$= \frac{\frac{V}{L}}{(s_2-s_1)}$$

$$B = (s-s_2) I(s) \big|_{s=s_2}$$

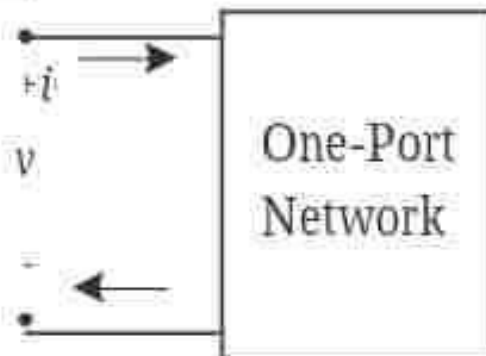
$$= \frac{\frac{V}{L}}{(s_2-s_1)} = -\frac{\frac{V}{L}}{(s_1-s_2)}$$

$$I(s) = \frac{\frac{V}{L}}{(s_1-s_2)} \left(\frac{1}{(s-s_1)} - \frac{1}{(s-s_2)} \right)$$

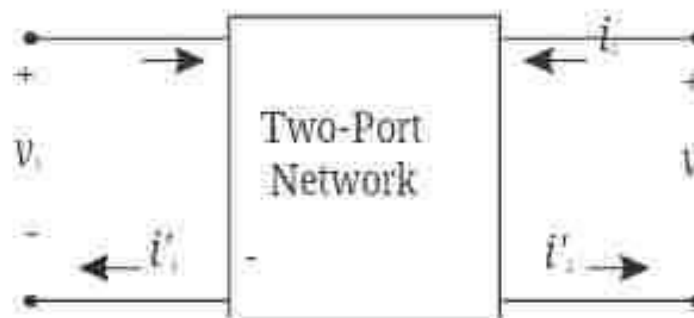
Taking the Inverse Laplace Transform

CHAPTER 8

TWO-PORT NETWORKS



- a) **One port network** is a two-terminal electrical network in which, current enters through one terminal and leaves through another terminal. Resistors, inductors and capacitors are the examples of one port network because each one has two terminals. One port network representation is shown in the following figure.
- b) A pair of terminals at which a signal (voltage or current) may enter or leave is called a port.
- c) A network having only one such pair of terminals is called a one-port network.
- d) No connections may be made to any other nodes internal to the network.
- e) By KCL, we therefore have $i_1 = i_2$



- **two port network** is a pair of two terminal electrical network in which, current enters through one terminal and leaves through another terminal of each port. Two port network representation is shown in the following figure. Type equation here.
- Two-port networks are used to describe the relationship between a pair of terminals
- The analysis methods we will discuss require the following conditions be met
 1. Linearity
 2. No independent sources inside the network
 3. No stored energy inside the network (zero initial conditions)
 4. $i_1 = i_1'$ and $i_2 = i_2'$

Two Port Network Parameters

There are various parameters needed to analyze a two port network. For examples, Z parameters, Y parameters, h parameters, g parameters, ABCD parameters etc.

Let us discuss these network parameters one by one to gain a better understanding of their application and uses.

Impedance Parameters

- Suppose the currents and voltages can be measured.
- Alternatively, if the circuit in the box is known, V_1 and V_2 can be calculated based on circuit analysis.
- Relationship can be written in terms of the impedance parameters.
- We can also calculate the impedance parameters after making two sets of measurements.

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$

$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

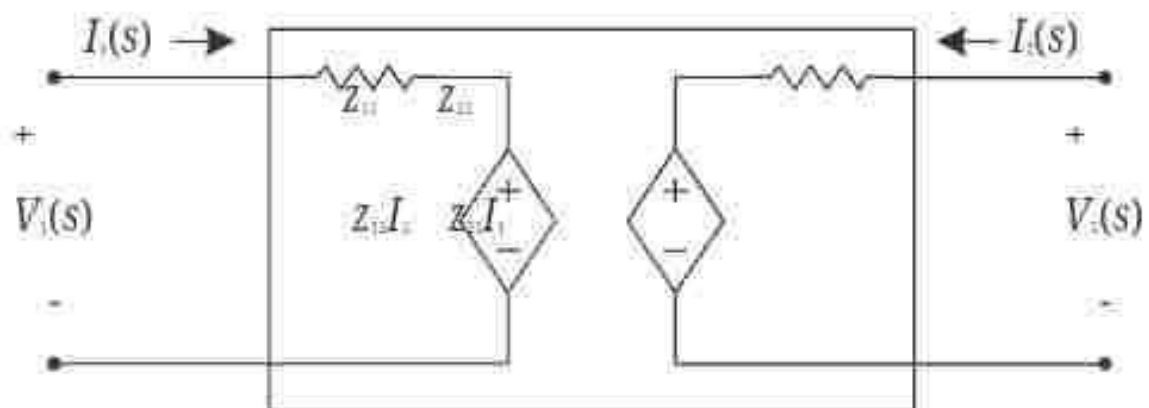
If the right port is an open circuit ($I_2=0$), then we can easily solve for two of the impedance parameters: Similarly by open circuiting left hand port ($I_1=0$) we can solve for the other two parameters.

⇒

$$Z_{11} = \text{input impedance} = \left. \frac{V_1}{I_1} \right|_{I_2=0} \quad Z_{21} = \text{forward transfer impedance} = \left. \frac{V_2}{I_1} \right|_{I_2=0}$$

$$Z_{12} = \text{reverse transfer impedance} = \left. \frac{V_1}{I_2} \right|_{I_1=0} \quad Z_{22} = \text{output impedance} = \left. \frac{V_2}{I_2} \right|_{I_1=0}$$

Impedance Parameter Equivalent

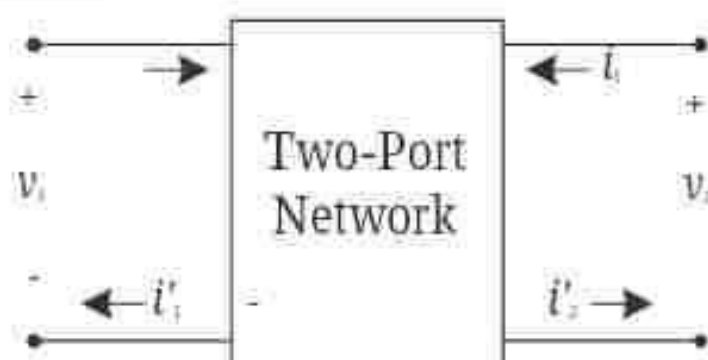


$$V_1 = Z_{11}I_1 + Z_{12}I_2$$

$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

- Once we know what the impedance parameters are, we can model the behavior of the two-port with an equivalent circuit.
- Notice the similarity to Th evenin and Norton equivalents

Admittance Parameters



$$I_1 = y_{11}V_1 + y_{12}V_2$$

$$I_2 = y_{21}V_1 + y_{22}V_2$$

$$Y_{11} = \text{input admittance} = \frac{I_1}{V_1} | V_2 = 0$$

$$Y_{12} = \text{forward transfer admittance} = \frac{I_1}{V_2} | V_1 = 0$$

$$Y_{22} = \text{output admittance} = \frac{I_2}{V_2} | V_1 = 0$$

$$Y_{21} = \text{reverse transfer admittance} = \frac{I_2}{V_1} | V_2 = 0$$

Hybrid Parameters

$$V_1 = h_{11}I_1 + h_{12}V_2$$

$$I_2 = h_{21}I_1 + h_{22}V_2$$

$$h_{11} = \text{input impedance} = \frac{V_1}{I_1} | V_2 = 0$$

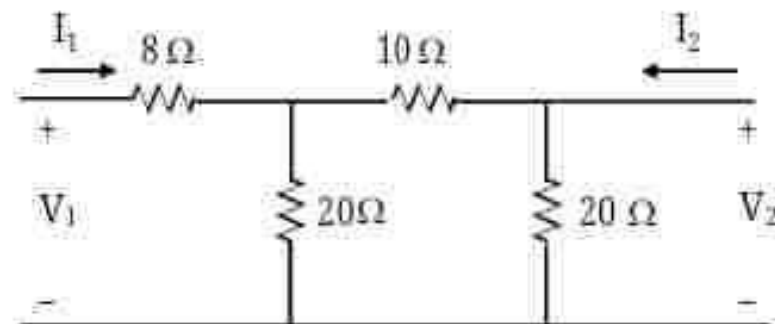
$$h_{21} = \text{forward current ratio} = \frac{I_2}{I_1} | V_2 = 0$$

$$h_{12} = \text{reverse voltage ratio} = \frac{V_1}{V_2} | I_1 = 0$$

$$h_{22} = \text{output admittance} = \frac{I_2}{V_2} | I_1 = 0$$

Example:

Given the following circuit. Determine the Z parameters.



$$Z_{11} = 8 + 20 \parallel 30 = 20 \Omega$$

$$Z_{22} = 20 \parallel 30 = 12 \Omega$$

$$Z_{12} = \frac{V_1}{I_2} \bigg|_{I_1=0}$$

$$V_1 = \frac{20 \times I_2 \times 20}{20 + 30} = 8 I_2 \quad \text{Therefore } Z_{12} = \frac{8 I_2}{I_2} = 8 \Omega = Z_{21}$$

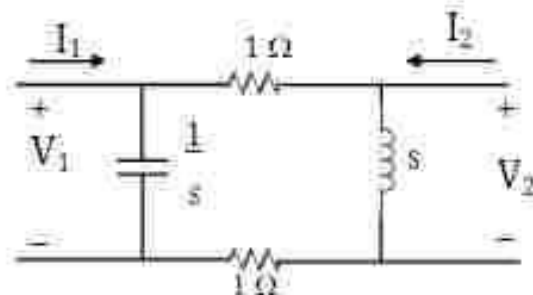
The Z parameter equations can be expressed in matrix form as follows.

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 20 & 8 \\ 8 & 12 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

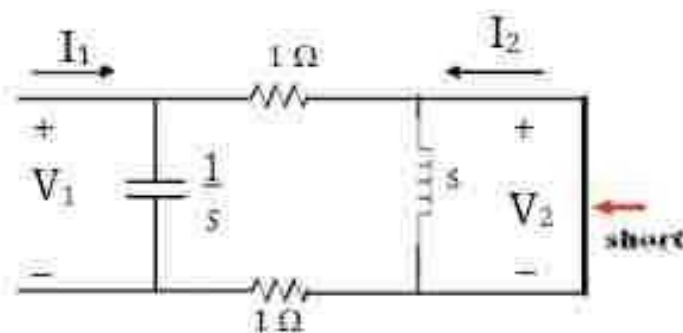
Example:

Given the following circuit. Determine the Y parameters.



$$I_1 = y_{11} V_1 + y_{12} V_2$$

$$I_2 = y_{21} V_1 + y_{22} V_2$$



To find y_{11}

$$V_1 = I_1 \left(\frac{2/s}{2 + 1/s} \right) = I_1 \left[\frac{2}{2s + 1} \right]$$

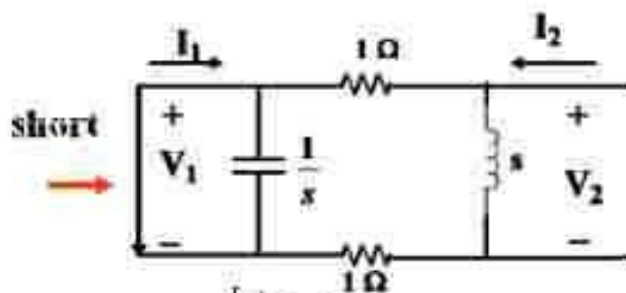
$$\text{so } y_{11} = \frac{I_1}{V_1} \quad y_{11} = \frac{I}{V_1} \bigg|_{V_2=0} = s + 0.5$$

To find y_{12} and y_{21} we reverse things and short V_1

$$y_{21} = \frac{I_2}{V_1} \bigg|_{V_1=0}$$

$$V_1 = -2I_2$$

$$y_{21} = \frac{I_2}{V_1} = 0.5 \text{ S}$$



$$y_{12} = \frac{I_1}{V_2} \bigg|_{V_1=0}$$

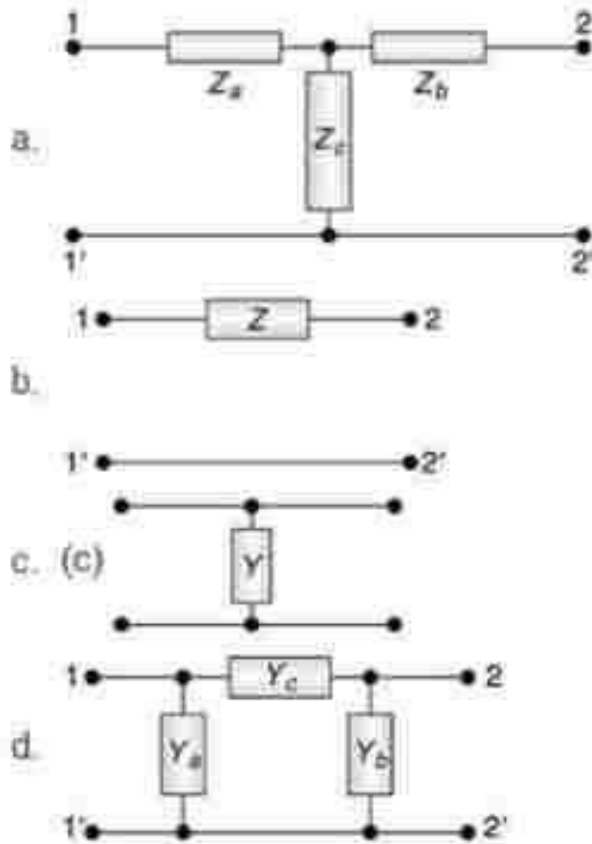
$$V_1 = -2I_1 \quad y_{12} = \frac{I_1}{V_2} = 0.5s$$

$$y_{22} = 0.5 + \frac{1}{s}$$

$$y_{22} = \frac{I_2}{V_2} \bigg|_{V_1=0} \quad V_2 = I_2 \frac{2s}{s+2} \quad y_{22} = 0.5 + \frac{1}{s}$$

Problem 1

Find the Z and Y parameter for the networks shown in figure.



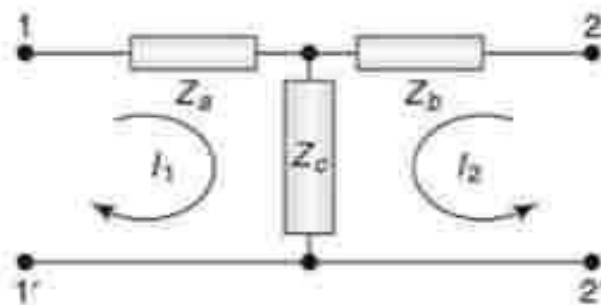
Solution

a. By KVL, $(Z_a + Z_c)I_1 + Z_c I_2 = V_1$

and $Z_c I_1 + (Z_b + Z_c)I_2 = V_2$

Thus, the Z-parameters are:

$$z_{11} = (Z_a + Z_c), z_{12} = z_{21} = Z_c, z_{22} = (Z_b + Z_c)$$



b. By KCL,

$$I_1 = \frac{V_1 - V_2}{Z} = \frac{1}{Z} V_1 - \frac{1}{Z} V_2$$

$$\text{and } I_2 = \frac{V_2 - V_1}{Z} = -\frac{1}{Z} V_1 + \frac{1}{Z} V_2$$

Thus, the y -parameters are,

$$y_{11} = \frac{1}{Z} = y_{22}, \quad y_{12} = y_{21} = -\frac{1}{Z}$$

Since, $\Delta y = y_{11}y_{22} - y_{12}y_{21} = 0$, the z -parameters do not exist for this network.



c. By KVL,

$$V_1 =$$

$$\frac{I_1 + I_2}{Y} = V_2 \quad \text{or, } V_1 = \left(\frac{1}{Y}\right) I_1 + \left(\frac{1}{Y}\right) I_2 \quad \text{and} \quad V_2 = \left(\frac{1}{Y}\right) I_1 + \left(\frac{1}{Y}\right) I_2$$

Thus, the z -parameters are,

$$z_{11} = z_{22} = \frac{1}{Y} = z_{12} = z_{21}$$

Since, $\Delta z = z_{11}z_{22} - z_{12}z_{21} = 0$, the y -parameters do not exist for this network.

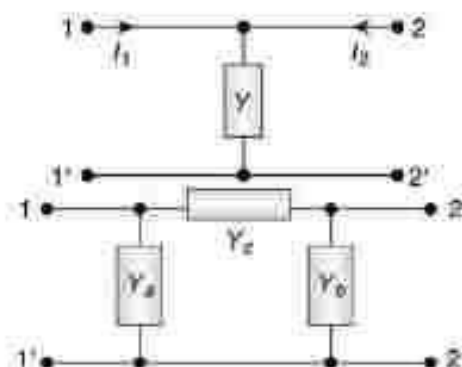
d. By KCL,

$$I_1 = Y_a V_1 + (V_1 - V_2) Y_c = V_1 (Y_a + Y_c) - V_2 Y_c$$

$$I_2 = Y_b V_2 + (V_2 - V_1) Y_c = -V_1 Y_c + V_2 (Y_b + Y_c)$$

Thus, the y -parameters are,

$$y_{11} = Y_a + Y_c; \quad y_{12} = y_{21} = -Y_c; \quad y_{22} = Y_b + Y_c$$



Problem 2

- a. The following equations give the voltages V_1 and V_2 at the two ports of a two port network, $V_1 = 5I_1 + 2I_2$, $V_2 = 2I_1 + I_2$.
A load resistance of $3\ \Omega$ is connected across port-2. Calculate the input impedance.
- b. The z-parameters of a two port network are $z_{11} = 5\ \Omega$, $z_{22} = 2\ \Omega$, $z_{12} = z_{21} = 3\ \Omega$. Load resistance of $4\ \Omega$ is connected across the output port. Calculate the input impedance.

Solution

- a. From the given equations,

$$V_1 = 5I_1 + 2I_2 \quad (i)$$

$$V_2 = 2I_1 + I_2 \quad (ii)$$

$$\text{At the output, } V_2 = -I_2 R_L = -3I_2$$

Putting this value in (ii),

$$-3I_2 = 2I_1 + I_2 \text{ or } I_2 = -I_1/2$$

$$\text{Putting in (i), } V_1 = 5I_1 + \left(\frac{-I_1}{2}\right) = 4I_1$$

$$\therefore \text{Input impedance, } Z_{in} = \frac{V_1}{I_1} = 4\ \Omega$$

$$\text{b. [Same as Prob. (a)] } Z_{in} = \frac{V_1}{I_1} = 3.5\ \Omega$$

Problem 3

Determine the h -parameters with the following data:

- with the output terminals short circuited, $V_1 = 25 \text{ V}$, $I_1 = 1 \text{ A}$, $I_2 = 2 \text{ A}$
- with the input terminals open circuited, $V_1 = 10 \text{ V}$, $V_2 = 50 \text{ V}$, $I_2 = 2 \text{ A}$

Solution

The h -parameter equations are,

$$V_1 = h_{11}I_1 + h_{12}V_2$$

$$I_2 = h_{21}I_1 + h_{22}V_2$$

- a. With output short circuited, $V_2 = 0$, given: $V_1 = 25 \text{ V}$, $I_1 = 1 \text{ A}$ and $I_2 = 2 \text{ A}$.

$$\therefore \quad \begin{cases} 25 = h_{11} \times 1 \\ 2 = h_{21} \times 1 \end{cases} \Rightarrow h_{11} = 25 \Omega, \text{ and } h_{21} = 2$$

- b. With input open circuited, $I_1 = 0$, given: $V_1 = 10 \text{ V}$, $V_2 = 50 \text{ V}$ and $I_2 = 2 \text{ A}$.

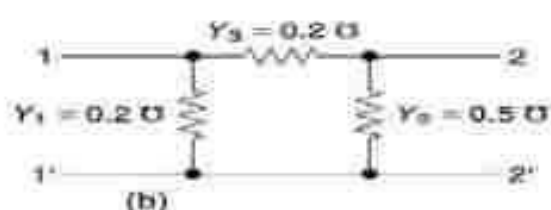
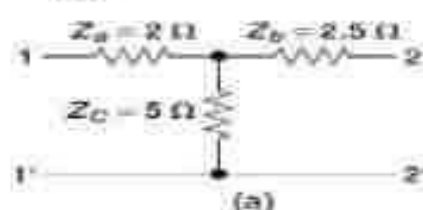
$$\therefore \quad \text{and} \quad \begin{cases} 10 = h_{12} \times 50 \\ 2 = h_{22} \times 50 \end{cases} \Rightarrow h_{12} = \frac{1}{5} = 0.2 \text{ and } h_{22} = \frac{1}{25} \Omega = 0.04 \Omega$$

Thus, the h -parameters are:

$$[h] = \begin{bmatrix} 25 \Omega & 0.2 \\ 2 & 0.04 \Omega^{-1} \end{bmatrix}$$

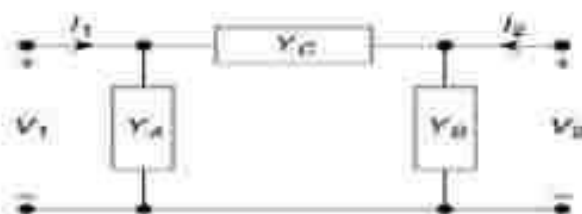
Problem 4

- Find the equivalent π -network for the T -network shown in the Fig. (a).
- Find the equivalent T -network for the π -network shown in the Fig. (b).



Solution

- a. Let the equivalent π -network have Y_C as the series admittance and Y_A and Y_B as the shunt admittances at port 1 and port 2, respectively.



Now, the z -parameters are given as:

$$z_{11} = (Z_A + Z_C) = 7 \Omega, z_{12} = z_{21} = Z_C = 5 \Omega, z_{22} = (Z_B + Z_C) = 7.5 \Omega$$

$$\therefore \Delta z = (7 \times 7.5 - 5 \times 5) = 27.5 \Omega^2$$

$$\therefore y_{11} = \frac{z_{22}}{\Delta z} = \frac{7.5}{27.5} \text{ S}$$

$$y_{12} = y_{21} = -\frac{z_C}{\Delta z} = -\frac{5}{27.5} \text{ S}$$

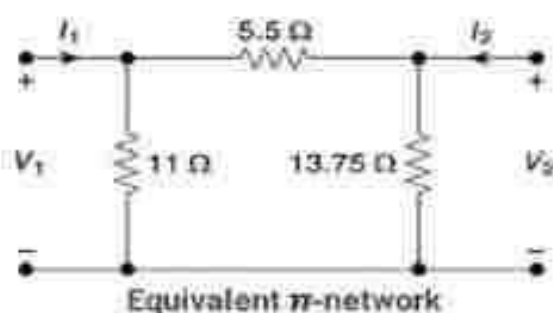
$$y_{22} = \frac{z_{11}}{\Delta z} = \frac{7}{27.5} \text{ S}$$

$$\therefore Y_A = (y_{11} + y_{12}) = \frac{2.5}{27.5} = \frac{1}{11} \text{ S}$$

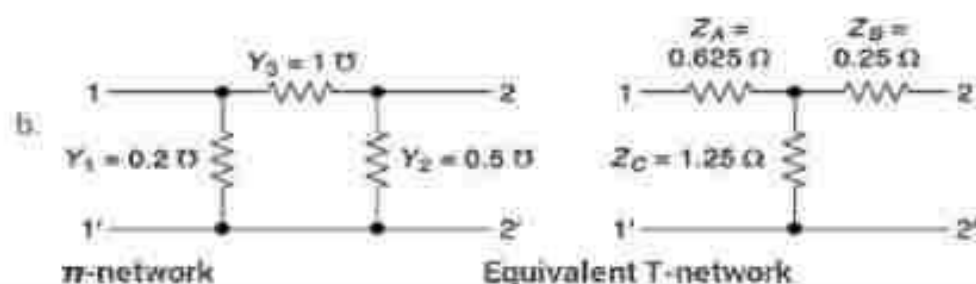
$$\therefore Y_B = (y_{22} + y_{12}) = \frac{2}{27.5} \text{ S}$$

$$\text{and } Y_C = -y_{21} = \frac{5}{27.5} = \frac{2}{11} \text{ S}$$

Thus, the impedances of the equivalent π -networks are:



$$\left. \begin{aligned} Z_A &= \frac{1}{Y_A} = 11 \Omega, \\ Z_B &= \frac{1}{Y_B} = 13.75 \Omega, \\ Z_C &= \frac{1}{Y_C} = 5.5 \Omega \end{aligned} \right\}$$



The y parameters,

$$y_{11} = 1.2 \text{ S}, y_{12} = y_{21} = -1 \text{ S}, \text{ and } y_{22} = 1.5 \text{ S}$$

$$\therefore \Delta y = (1.2 \times 1.5 - 1) = 0.8$$

$$\therefore z_{11} = \frac{y_{22}}{\Delta y} = \frac{1.5}{0.8} \Omega, z_{12} = z_{21} = -\frac{y_{12}}{\Delta y} = \frac{1}{0.8} \Omega, z_{22} = \frac{y_{11}}{\Delta y} = \frac{1.2}{0.8} \Omega$$

$$\left. \begin{aligned} Z_A &= (z_{11} - z_{12}) = \frac{0.5}{0.8} = 0.625 \Omega \\ Z_B &= (z_{22} - z_{12}) = \frac{0.2}{0.8} = 0.25 \Omega \\ Z_C &= z_{12} = \frac{1}{0.8} = 1.25 \Omega \end{aligned} \right\}$$



CHAPTER 9

LOW PASS FILTER INTRODUCTION

Basically, an electrical filter is a circuit that can be designed to modify, reshape or reject all unwanted frequencies of an electrical signal and accept or pass only those signals wanted by the circuit's designer. In other words they "filter-out" unwanted signals and an ideal filter will separate and pass sinusoidal input signals based upon their frequency.

In low frequency applications (up to 100kHz), passive filters are generally constructed using simple RC (Resistor-Capacitor) networks, while higher frequency filters (above 100kHz) are usually made from RLC (Resistor-Inductor-Capacitor) components.

Passive Filters are made up of passive components such as resistors, capacitors and inductors and have no amplifying elements (transistors, op-amps, etc) so have no signal gain, therefore their output level is always less than the input.

Filters are so named according to the frequency range of signals that they allow to pass through them, while blocking or "attenuating" the rest. The most commonly used filter designs are the:

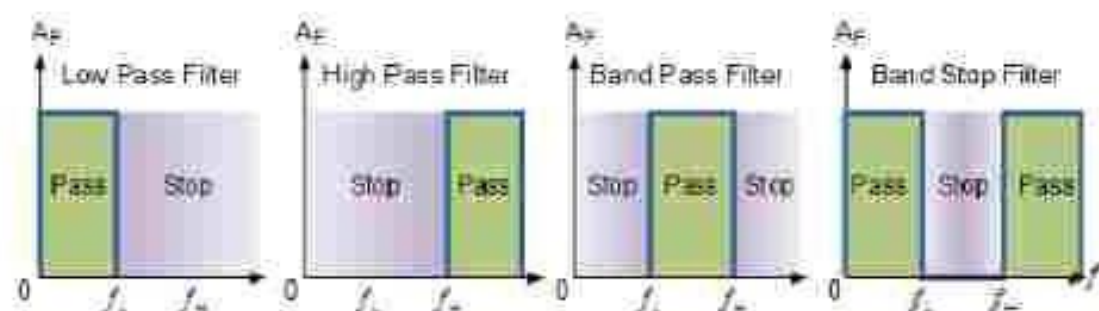
- 1. The Low Pass Filter – the low pass filter only allows low frequency signals from 0Hz to its cut-off frequency, f_c point to pass while blocking those any higher.
- 2. The High Pass Filter – the high pass filter only allows high frequency signals from its cut-off frequency, f_c point and higher to infinity to pass through while blocking those any lower.
- 3. The Band Pass Filter – the band pass filter allows signals falling within a certain frequency band setup between two points to pass through while blocking both the lower and higher frequencies either side of this frequency band.
- 4 Band Stop Filter - It is so called *band-elimination*, *band-reject*, or *notch* filters; this kind of filter passes all frequencies above and below a particular range set by the component values.

Simple First-order passive filters (1st order) can be made by connecting together a single resistor and a single capacitor in series across an input signal, (V_{in}) with the output of the filter, (V_{out}) taken from the junction of these two components. Depending on which way around we connect the resistor and the capacitor with regards to the output signal determines the type of filter construction resulting in either a Low Pass Filter or a High Pass Filter.

As the function of any filter is to allow signals of a given band of frequencies to pass unaltered while attenuating or weakening all others those are not

wanted, we can define the amplitude response characteristics of an ideal filter by using an ideal frequency response curve of the four basic filter types as shown.

IDEAL FILTER RESPONSE CURVES



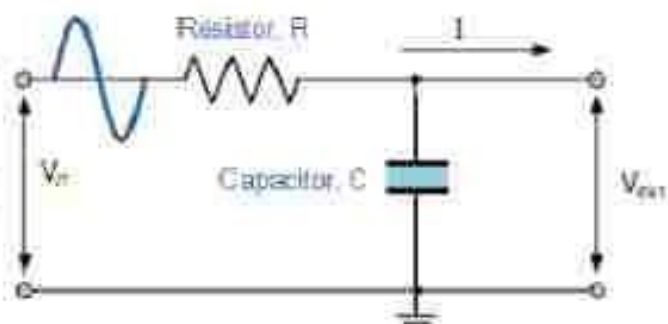
A Low Pass Filter can be a combination of capacitance, inductance or resistance intended to produce high attenuation above a specified frequency and little or no attenuation below that frequency. The frequency at which the transition occurs is called the "cutoff" frequency. The simplest low pass filters consist of a resistor and capacitor but more sophisticated low pass filters have a combination of series inductors and parallel capacitors. In this tutorial we will look at the simplest type, a passive two component RC low pass filter.

THE LOW PASS FILTER

A simple passive RC Low Pass Filter or LPF, can be easily made by connecting together in series a single Resistor with a single Capacitor as shown below. In this type of filter arrangement the input signal (V_{in}) is applied to the series combination (both the Resistor and Capacitor together) but the output signal (V_{out}) is taken across the capacitor only. This type of filter is known generally as a "first-order filter" or "one-pole filter", why first-order or single-pole?, because it has only "one" reactive component, the capacitor, in the circuit.

RC LOW PASS FILTER CIRCUIT

As mentioned previously in the Capacitive Reactance tutorial, the reactance of a capacitor varies inversely with frequency, while the value of the resistor remains constant as the frequency changes. At low frequencies the capacitive reactance, (X_c) of the capacitor will be very large compared to the resistive value of the resistor, R and as a result the voltage across the capacitor, V_c will also be large while the voltage drop across the resistor, V_r will be much lower. At high frequencies the reverse is true with V_c being small and V_r being large.



While the circuit above is that of an RC Low Pass Filter circuit, it can also be classed as a frequency variable potential divider circuit similar to the one we looked

at in the Resistors tutorial. In that tutorial we used the following equation to calculate the output voltage for two single resistors connected in series,

$$V_{out} = V_{in} \times \frac{R_2}{R_1 + R_2}$$

where $R_1 + R_2 = R_T$, the total resistance of the circuit

We also know that the capacitive reactance of a capacitor in an AC circuit is given as;

$$X_C = \frac{1}{2\pi fC} \text{ in Ohm's}$$

Opposition to current flow in an AC circuit is called impedance, symbol Z and for a series circuit consisting of a single resistor in series with a single capacitor, the circuit impedance is calculated as:

$$Z = \sqrt{R^2 + X_C^2}$$

Then by substituting our equation for impedance above into the resistive potential divider equation gives us;

RC POTENTIAL DIVIDER EQUATION

$$V_{out} = V_{in} \times \frac{X_C}{\sqrt{R^2 + X_C^2}} = V_{in} \frac{X_C}{Z}$$

So, by using the potential divider equation of two resistors in series and substituting for impedance we can calculate the output voltage of an RC Filter for any given frequency.

LOW PASS FILTER EXAMPLE

A Low Pass Filter circuit consisting of a resistor of $4k7\Omega$ in series with a capacitor of $47nF$ is connected across a $10v$ sinusoidal supply. Calculate the output voltage (V_{out}) at a frequency of $100Hz$ and again at frequency of $10,000Hz$ or $10kHz$.

Voltage Output at a Frequency of 100Hz.

$$X_c = \frac{1}{2\pi f C} = \frac{1}{2\pi \times 100 \times 47 \times 10^{-9}} = 33,863\Omega$$

$$V_{OUT} = V_{IN} \times \frac{X_c}{\sqrt{R^2 + X_c^2}} = 10 \times \frac{33863}{\sqrt{4700^2 + 33863^2}} = 9.9v$$

Voltage Output at a Frequency of 10,000Hz (10kHz).

$$X_c = \frac{1}{2\pi f C} = \frac{1}{2\pi \times 10,000 \times 47 \times 10^{-9}} = 338.6\Omega$$

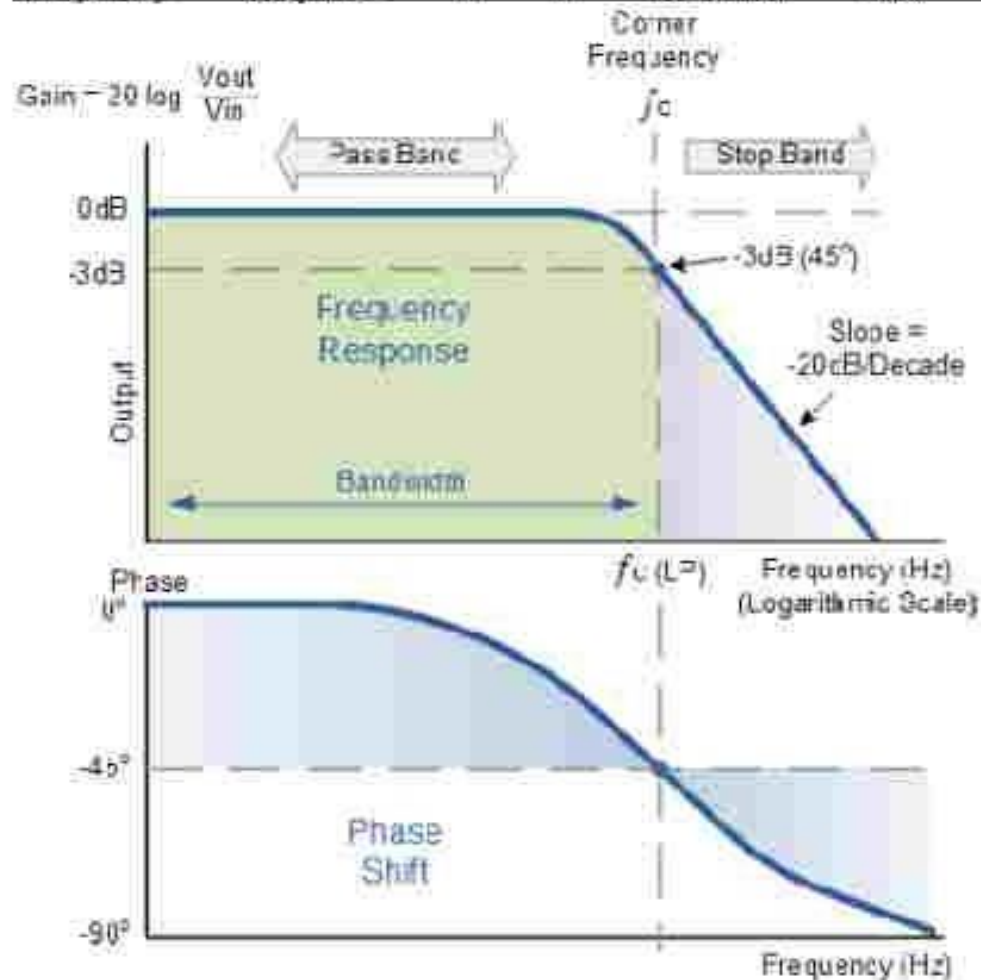
$$V_{OUT} = V_{IN} \times \frac{X_c}{\sqrt{R^2 + X_c^2}} = 10 \times \frac{338.6}{\sqrt{4700^2 + 338.6^2}} = 0.718v$$

FREQUENCY RESPONSE

We can see from the results above that as the frequency applied to the RC network increases from $100Hz$ to 10 kHz , the voltage dropped across the capacitor and therefore the output voltage (V_{out}) from the circuit decreases from $9.9v$ to $0.718v$.

By plotting the networks output voltage against different values of input frequency, the Frequency Response Curve or Bode Plot function of the low pass filter circuit can be found, as shown below.

Frequency Response of a 1st-order Low Pass Filter



The Bode Plot shows the Frequency Response of the filter to be nearly flat for low frequencies and the entire input signal is passed directly to the output, resulting in a gain of nearly 1, called unity, until it reaches its Cut-off Frequency point (f_c). This is because the reactance of the capacitor is high at low frequencies and blocks any current flow through the capacitor.

After this cut-off frequency point the response of the circuit decreases to zero at a slope of -20dB/ Decade or (-6dB/Octave) "roll-off". Note that the angle of the slope, this -20dB/ Decade roll-off will always be the same for any RC combination.

Any high frequency signals applied to the low pass filter circuit above this cut-off frequency point will become greatly attenuated, that is they rapidly decrease. This happens because at very high frequencies the reactance of the capacitor becomes so low that it gives the effect of a short circuit condition on the output terminals resulting in zero output.

Then by carefully selecting the correct resistor-capacitor combination, we can create a RC circuit that allows a range of frequencies below a certain value to pass through the circuit unaffected while any frequencies applied to the circuit above this cut-off point to be attenuated, creating what is commonly called a Low Pass Filter.

For this type of "Low Pass Filter" circuit, all the frequencies below this cut-off, f_c point that are unaltered with little or no attenuation and are said to be in the filter's Pass band zone. This pass band zone also represents the Bandwidth of the filter. Any signal frequencies above this point cut-off point are generally said to be in the filter's Stop band zone and they will be greatly attenuated.

This "Cut-off", "Corner" or "Breakpoint" frequency is defined as being the frequency point where the capacitive reactance and resistance are equal, $R = X_c = 4k7\Omega$. When this occurs the output signal is attenuated to 70.7% of the input signal value or -3dB ($20 \log (V_{out}/V_{in})$) of the input. Although $R = X_c$, the output is not half of the input signal. This is because it is equal to the vector sum of the two and is therefore 0.707 of the input.

As the filter contains a capacitor, the Phase Angle (Φ) of the output signal LAGS behind that of the input and at the -3dB cut-off frequency (f_c) and is -45° out of phase. This is due to the time taken to charge the plates of the capacitor as the input voltage changes, resulting in the output voltage (the voltage across the capacitor) "lagging" behind that of the input signal. The higher the input frequency applied to the filter the more the capacitor lags and the circuit becomes more and more "out of phase".

The cut-off frequency point and phase shift angle can be found by using the following equation:

CUT-OFF FREQUENCY AND PHASE SHIFT

$$f_c = \frac{1}{2\pi RC} = \frac{1}{2\pi \times 4700 \times 47 \times 10^{-9}} = 720\text{Hz}$$

$$\text{Phase Shift } \phi = -\arctan (2\pi f RC)$$

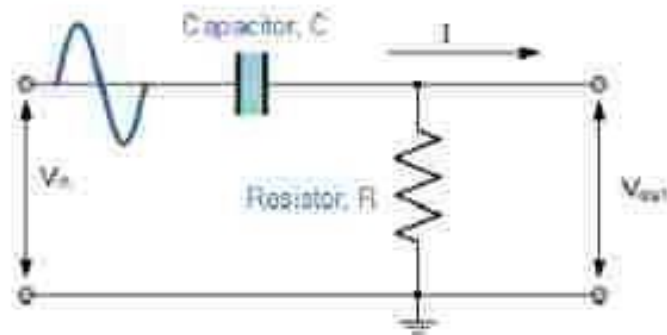
Then for our simple example of a "Low Pass Filter" circuit above, the cut-off frequency (f_c) is given as 720Hz with an output voltage of 70.7% of the input voltage value and a phase shift angle of -45°.

HIGH PASS FILTERS

A High Pass Filter or HPF, is the exact opposite to that of the previously seen Low Pass filter circuit, as now the two components have been interchanged with the output signal (V_{out}) being taken from across the resistor as shown.

Where as the low pass filter only allowed signals to pass below its cut-off frequency point, f_c , the passive high pass filter circuit as its name implies, only passes signals above the selected cut-off point, f_c eliminating any low frequency signals from the waveform. Consider the circuit below.

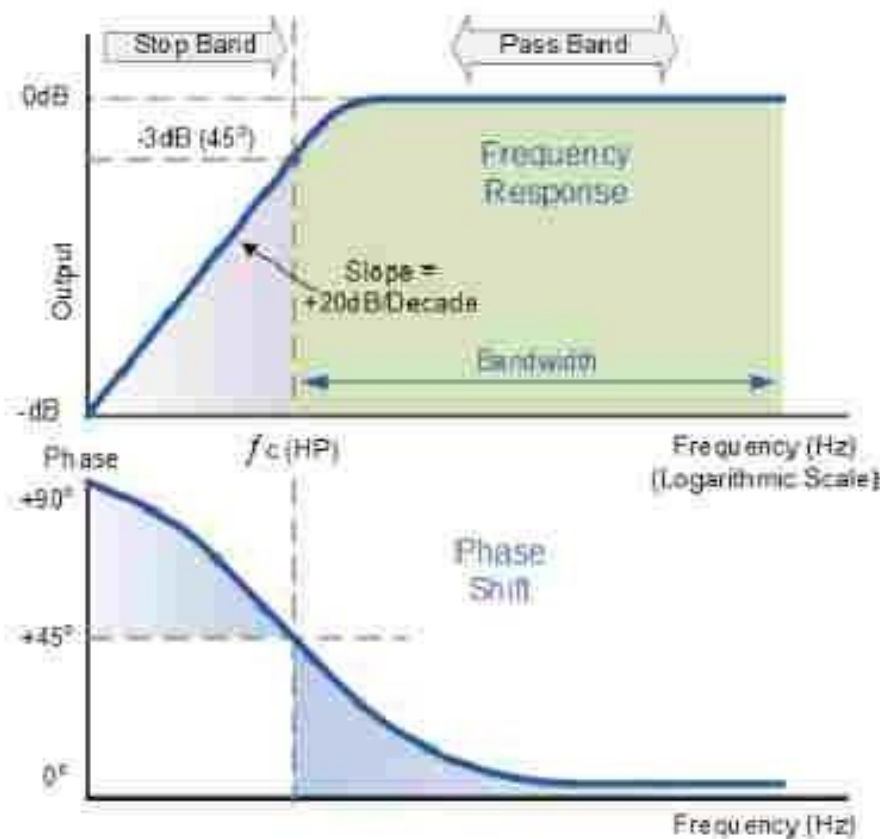
THE HIGH PASS FILTER CIRCUIT



In this circuit arrangement, the reactance of the capacitor is very high at low frequencies so the capacitor acts like an open circuit and blocks any input signals at V_{in} until the cut-off frequency point (f_c) is reached. Above this cut-off frequency point the reactance of the capacitor has reduced sufficiently as to now act more like a short circuit allowing the entire input signal to pass directly to the output as shown below in the High Pass Frequency Response Curve.

FREQUENCY RESPONSE OF A 1ST ORDER HIGH PASS FILTER

$$\text{Gain (dB)} = 20 \log \frac{V_{out}}{V_{in}}$$



The Bode Plot or Frequency Response Curve above for a High Pass filter is the exact opposite to that of a low pass filter. Here the signal is attenuated or damped at low frequencies with the output increasing at +20dB/Decade (6dB/Octave) until the frequency reaches the cut-off point (f_c) where again $R = X_c$. It has a response curve that extends down from infinity to the cut-off frequency, where the output voltage amplitude is $1/\sqrt{2} \approx 70.7\%$ of the input signal value or -3dB (20 log (Vout/Vin)) of the input value.

Also we can see that the phase angle (Φ) of the output signal LEADS that of the input and is equal to +45° at frequency f_c . The frequency response curve for a high pass filter implies that the filter can pass all signals out to infinity. However in practice, the high pass filter response does not extend to infinity but is limited by the electrical characteristics of the components used.

The cut-off frequency point for a first order high pass filter can be found using the same equation as that of the low pass filter, but the equation for the phase shift is modified slightly to account for the positive phase angle as shown below.

CUT-OFF FREQUENCY AND PHASE SHIFT

$$f_c = \frac{1}{2\pi RC}$$

$$\text{Phase Shift } \phi = \arctan \frac{1}{2\pi fRC}$$

The circuit gain, A_v which is given as Vout/Vin (magnitude) and is calculated as:

$$A_v = \frac{V_{OUT}}{V_{IN}} = \frac{R}{\sqrt{R^2 + X_C^2}} = \frac{R}{Z}$$

at low f : $X_c \rightarrow \infty$, $V_{out} = 0$

at high f : $X_c \rightarrow 0$, $V_{out} = V_{in}$

HIGH PASS FILTER EXAMPLE:

Calculate the cut-off or "breakpoint" frequency (f_c) for a simple high pass filter consisting of an 82pF capacitor connected in series with a 240kΩ resistor.

$$f_c = \frac{1}{2\pi RC} = \frac{1}{2\pi \times 240,000 \times 82 \times 10^{-12}} = 8,087 \text{ Hz or } 8 \text{ kHz}$$

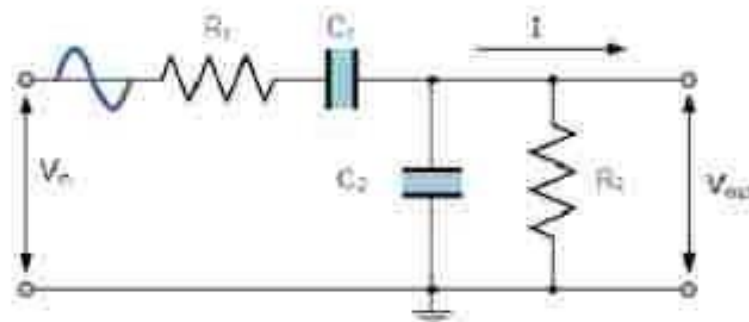
BAND PASS FILTERS

The cut-off frequency or f_c point in a simple RC passive filter can be accurately controlled using just a single resistor in series with a non-polarized capacitor, and depending upon which way around they are connected either a low pass or a high pass filter is obtained.

One simple use for these types of Passive Filters is in audio amplifier applications or circuits such as in loudspeaker crossover filters or pre-amplifier tone controls. Sometimes it is necessary to only pass a certain range of frequencies that do not begin at 0Hz, (DC) or end at some high frequency point but are within a certain frequency band, either narrow or wide.

By connecting or "cascading" together a single Low Pass Filter circuit with a High Pass Filter circuit, we can produce another type of passive RC filter that passes a selected range or "band" of frequencies that can be either narrow or wide while attenuating all those outside of this range. This new type of passive filter arrangement produces a frequency selective filter known commonly as a Band Pass Filter or BPF for short.

BAND PASS FILTER CIRCUIT



Unlike a low pass filter that only pass signals of a low frequency range or a high pass filter which pass signals of a higher frequency range, a Band Pass Filters passes signals within a certain "band" or "spread" of frequencies without distorting the input signal or introducing extra noise. This band of frequencies can be any width and is commonly known as the filters Bandwidth.

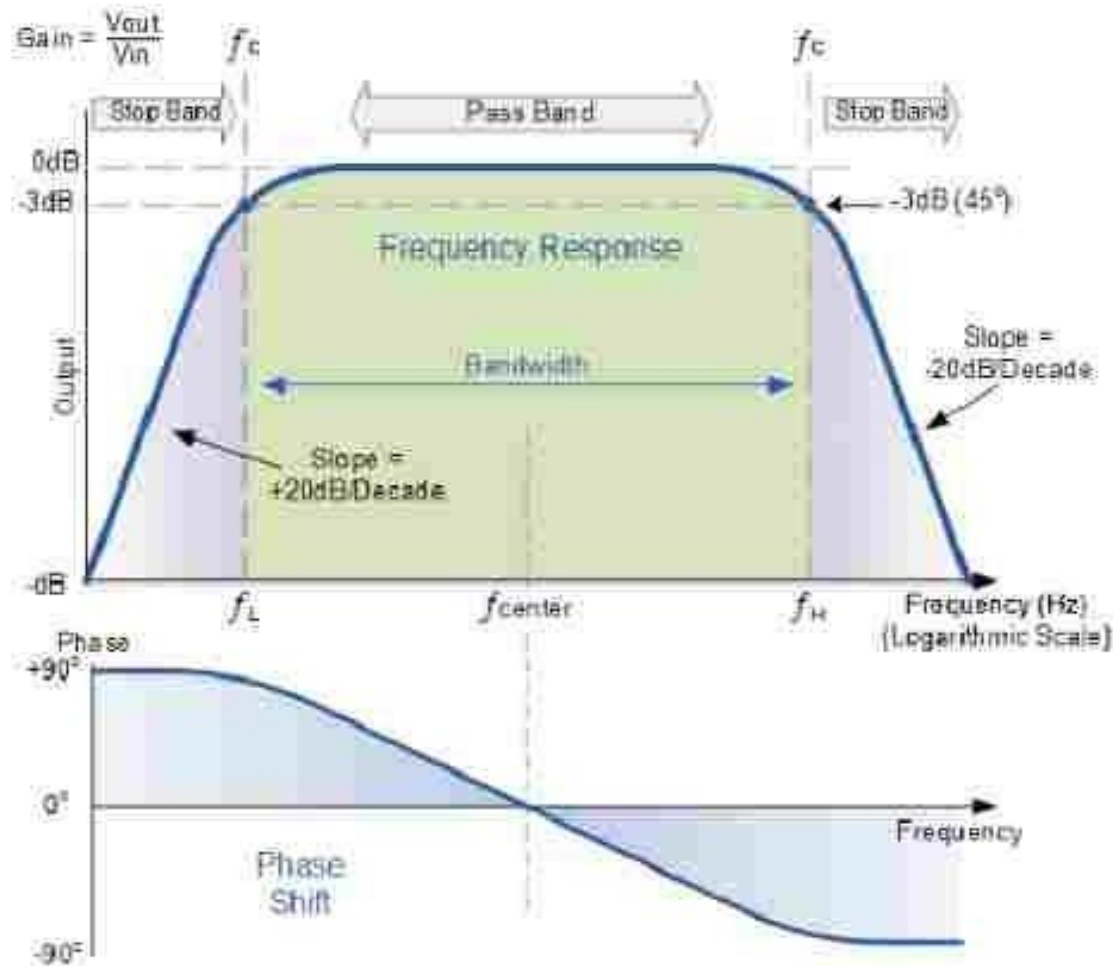
Bandwidth is commonly defined as the frequency range that exists between two specified frequency cut-off points (f_c), that are 3dB below the maximum centre or resonant peak while attenuating or weakening the others outside of these two points.

Then for widely spread frequencies, we can simply define the term "bandwidth", BW as being the difference between the lower cut-off frequency ($f_{c_{lower}}$) and the higher cut-off frequency ($f_{c_{higher}}$) points. In other words, $BW = f_H - f_L$. Clearly for a pass band filter to function correctly, the cut-off frequency of the low pass filter must be higher than the cut-off frequency for the high pass filter.

The "ideal" Band Pass Filter can also be used to isolate or filter out certain frequencies that lie within a particular band of frequencies, for example, noise

cancellation. Band pass filters are known generally as second-order filters, (two-pole) because they have "two" reactive component, the capacitors, within their circuit design. One capacitor in the low pass circuit and another capacitor in the high pass circuit.

Frequency Response of a 2nd Order Band Pass Filter.



The Bode Plot or frequency response curve above shows the characteristics of the band pass filter. Here the signal is attenuated at low frequencies with the output increasing at a slope of +20dB/Decade (6dB/Octave) until the frequency reaches the "lower cut-off" point f_L . At this frequency the output voltage is again $1/\sqrt{2} = 70.7\%$ of the input signal value or -3dB ($20 \log (V_{\text{out}}/V_{\text{in}})$) of the input.

The output continues at maximum gain until it reaches the "upper cut-off" point f_H where the output decreases at a rate of -20dB/Decade (6dB/Octave) attenuating any high frequency signals. The point of maximum output gain is generally the geometric mean of the two -3dB value between the lower and upper cut-off points and is called the "Centre Frequency" or "Resonant Peak" value f_r . This geometric mean value is calculated as being $f_r^2 = f_{(\text{upper})} \times f_{(\text{lower})}$.

A band pass filter is regarded as a second-order (two-pole) type filter because it has "two" reactive components within its circuit structure, then the phase angle will be twice that of the previously seen first-order filters, i.e., 180°. The phase

angle of the output signal LEADS that of the input by $+90^\circ$ up to the centre or resonant frequency, fr point where it becomes "zero" degrees (0°) or "in-phase" and then changes to LAG the input by -90° as the output frequency increases.

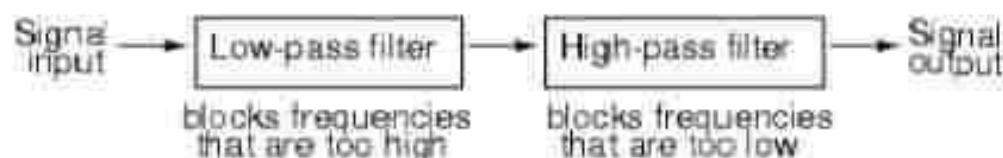
The upper and lower cut-off frequency points for a band pass filter can be found using the same formula as that for both the low and high pass filters, For example,

$$f_c = \frac{1}{2\pi RC} \text{ Hz}$$

Then clearly, the width of the pass band of the filter can be controlled by the positioning of the two cut-off frequency points of the two filters,

Band Pass Filter Example

A second-order band pass filter is to be constructed using RC components that will only allow a range of frequencies to pass above 1kHz (1,000Hz) and below 30kHz (30,000Hz). Assuming that both the resistors have values of 10k Ω 's, calculate the values of the two capacitors required.



The High Pass Filter Stage

The value of the capacitor C1 required to give a cut-off frequency f_L of 1kHz with a resistor value of 10k Ω is calculated as:

$$C = \frac{1}{2\pi f_c R} = \frac{1}{2\pi \times 1,000 \times 10,000} = 15.8 \text{ nF}$$

Then, the values of R1 and C1 required for the high pass stage to give a cut-off frequency of 1.0kHz i.e: R1 = 10k Ω 's and C1 = 15nF.

The Low Pass Filter Stage

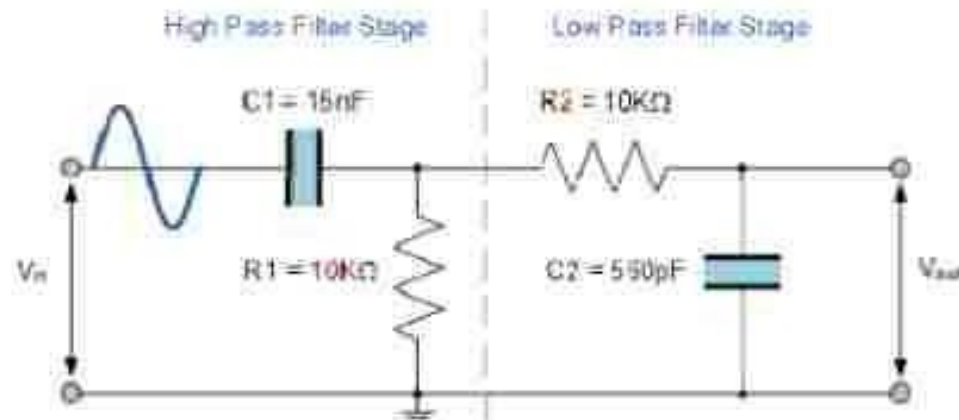
The value of the capacitor C2 required to give a cut-off frequency f_H of 30kHz with a resistor value of 10k Ω is calculated as:

$$C = \frac{1}{2\pi f_c R} = \frac{1}{2\pi \times 30,000 \times 10,000} = 510 \text{ pF}$$

Then, the values of R_2 and C_2 required for the low pass stage to give a cut-off frequency of 30kHz are, $R = 10k\Omega$'s and $C = 510pF$. However, the nearest preferred value of the calculated capacitor value of 510pF is 560pF so this is used instead.

With the values of both the resistances R_1 and R_2 given as $10k\Omega$, and the two values of the capacitors C_1 and C_2 found for the high pass and low pass filters as 15nF and 560pF respectively, then the circuit for our simple passive Band Pass Filter is given as.

Completed Band Pass Filter Circuit



Band Pass Filter Resonant Frequency

We can also calculate the "Resonant" or "Centre Frequency" (f_r) point of the band pass filter where the output gain is at its maximum or peak value. This peak value is not the arithmetic average of the upper and lower -3dB cut-off points as you might expect but is in fact the "geometric" or mean value. This geometric mean value is calculated as being $f_r = \sqrt{f_{c(upper)} \times f_{c(lower)}}$ for example:

Centre Frequency Equation

$$f_r = \sqrt{f_L \times f_H}$$

- Where, f_r is the resonant or centre frequency
- f_L is the lower -3dB cut-off frequency point
- f_H is the upper -3db cut-off frequency point

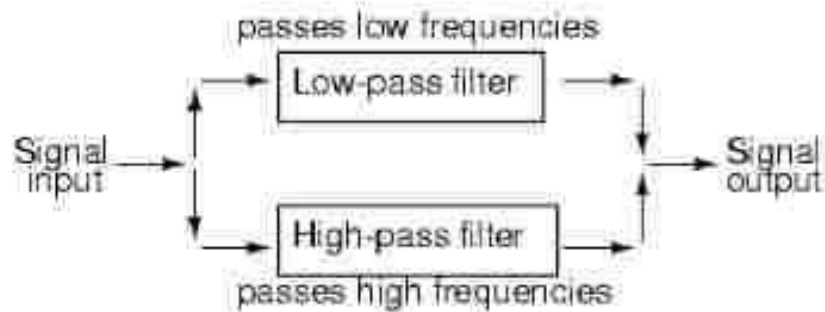
And in our simple example above, the calculated cut-off frequencies were found to be $f_L = 1,060$ Hz and $f_H = 28,420$ Hz using the filter values.

Then by substituting these values into the above equation gives a central resonant frequency of:

$$f_r = \sqrt{f_L \times f_H} = \sqrt{1,060 \times 28,420} = 5,48 \text{ kHz}$$

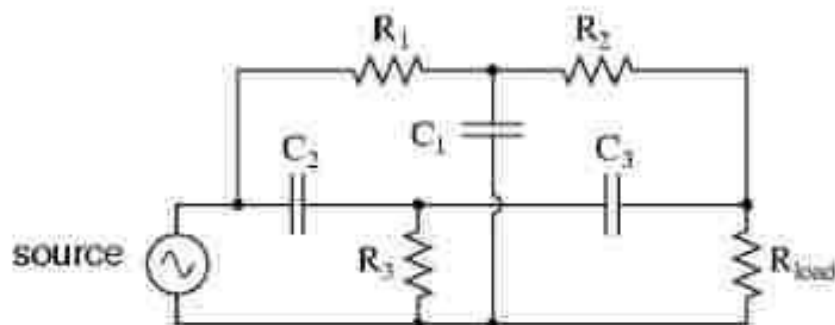
Band-stop filters

It is so-called *band-elimination*, *band-reject*, or *notch* filters; this kind of filter passes all frequencies above and below a particular range set by the component values. Not surprisingly, it can be made out of a low-pass and a high-pass filter, just like the band-pass design, except that this time we connect the two filter sections in parallel with each other instead of in series. (Figure below)



System level block diagram of a band-stop filter.

Constructed using two capacitive filter sections, it looks something like (Figure below).



Butterworth Filter

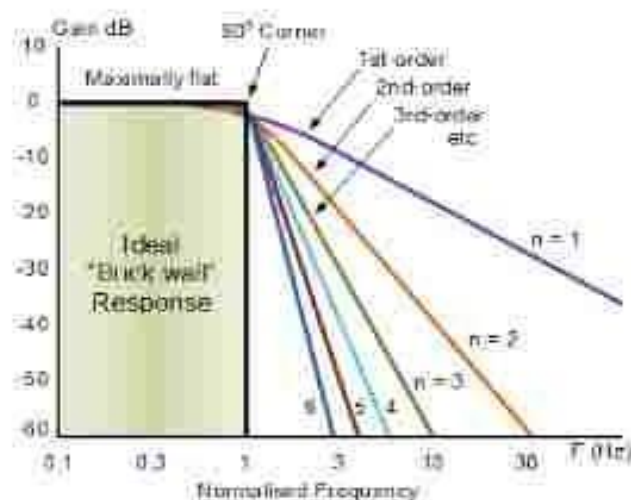
A Butterworth filter is a type of signal processing filter designed to have a frequency response as flat as possible in the passband. Hence the Butterworth filter is also known as “maximally flat magnitude filter”. It was invented in 1930 by the British engineer and physicist Stephen Butterworth in his paper titled “*On the Theory of Filter Amplifiers*”. The frequency response of the Butterworth filter is flat in the passband (i.e. a bandpass filter) and roll-offs towards zero in the stopband. The rate of roll-off response depends on the order of the filter. The number of reactive elements used in the filter circuit will decide the order of the filter.

The inductor and capacitor are reactive elements used in filters. But in the case of Butterworth filter only capacitors are used. So, the number of capacitors will decide the order of the filter.

Here, we will discuss the Butterworth filter with a low pass filter. Similarly, the high pass filter can be designed by just changing the position of resistance and capacitance.

Butterworth Low Pass Filter Design

While designing the filter, the designer tries to achieve a response near to the ideal filter. It is very difficult to match results with the exact ideal characteristic. We need to use complex higher-order If you increase the order of the filter, the number of cascade stages with the filter is also increased. But in practice, we cannot achieve Butterworth’s ideal frequency response. Because it produces excessive ripple in the passband. In Butterworth filter, mathematically it is possible to get flat frequency response from 0 Hz to the cut-off frequency at -3dB with no ripple. If the frequency is more than the cut-off frequency, it will roll-off towards zero with the rate of -20 dB/decade for the first-order filter. If you increase the order of the filter, the rate of a roll-off period is also increased. And for second-order, it is -40 dB/decade. The quality factor for the Butterworth filter is 0.707. The below figure shows the frequency response of the Butterworth filter for various orders of the filter



Frequency Response of Butterworth Filter The generalized form of frequency response for nth-order Butterworth low-pass filter is;

$$H(j\omega) = \frac{1}{\sqrt{1 + \epsilon^2 (\frac{\omega}{\omega_c})^{2n}}}$$

Where,

n = order of the filter,

ω = operating frequency (passband frequency) of circuit

ω_c = Cut-off frequency

ϵ = maximum passband gain = A_{max}

The below equation is used to find the value of ϵ .

$$H_1 = \frac{H_0}{\sqrt{1 + \epsilon^2}}$$

Where,

H_1 = minimum passband gain

H_0 = maximum passband gain

First-order Lowpass Butterworth Filter

The lowpass filter is a filter that allows the signal with the frequency is lower than the cutoff frequency and attenuates the signals with the frequency is more than cutoff frequency. In the first-order filter, the number of reactive components is only one. The below figure shows the circuit diagram of the first-order lowpass Butterworth filter.

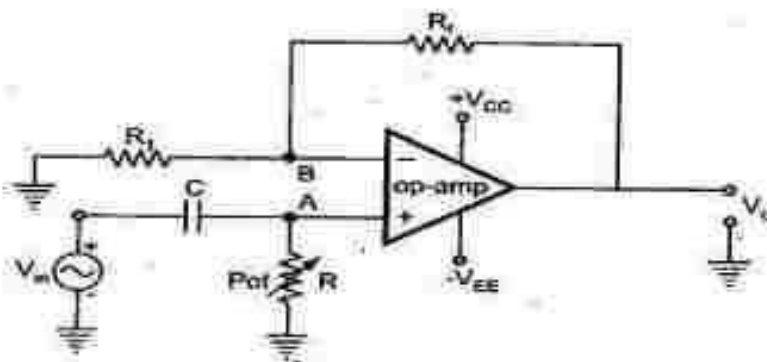


Fig. 2.79 First order high pass Butterworth filter

The low pass Butterworth filter is an active Low pass filter as it consists of the op-amp. This op-amp operates on non-inverting mode. Hence, the gain of the filter will decide by the resistor R_1 and R_f . And the cutoff frequency decides by R and C .

Now, if you apply the voltage divider rule at point V_a and find the voltage across a capacitor. It is given as;

$$V_a = \frac{-jX_C}{R - jX_C} V_{in}$$

$$V_a = \frac{-j(\frac{1}{2\pi fC})}{R - j(\frac{1}{2\pi fC})} V_{in}$$

$$V_a = \frac{-j}{2\pi fRC - j} V_{in}$$

$$V_a = \frac{V_{in}}{1 - \frac{2\pi fRC}{j}}$$

$$V_a = \frac{V_{in}}{1 + j2\pi fRC}$$

Because of the non-inverting configuration of an op-amp,

$$V_0 = \left(1 + \frac{R_f}{R_1}\right) V_a$$

$$V_0 = \left(1 + \frac{R_f}{R_1}\right) \frac{V_{in}}{1 + j2\pi fRC}$$

$$\frac{V_0}{V_a} = \frac{A_f}{1 + j\frac{f}{f_c}}$$

WHERE

$$A_f = 1 + \frac{R_f}{R_1}$$

A_f = Gain of filter in Passband

$$f_c = \frac{1}{2\pi RC}$$

f_c = Cutoff Frequency

f = Operating Frequency

$$\frac{V_o}{V_a} = \left| \frac{V_o}{V_a} \right| \angle \phi$$

$$\left| \frac{V_o}{V_a} \right| = \frac{A_f}{\sqrt{1 + \left(\frac{f}{f_c} \right)^2}}$$

$$\phi = -\tan^{-1} \left(\frac{f}{f_c} \right)$$

1. At very low frequency, $f \ll f_c$

$$\left| \frac{V_o}{V_a} \right| \approx A_f (\text{Constant})$$

2. At cutoff frequency, $f = f_c$

$$\left| \frac{V_o}{V_a} \right| = \frac{A_f}{\sqrt{2}} = 0.707 A_f$$

3. At high frequency, $f > f_c$

$$\left| \frac{V_o}{V_a} \right| < A_f$$

The below figure shows the frequency response of first-order lowpass Butterworth filter.

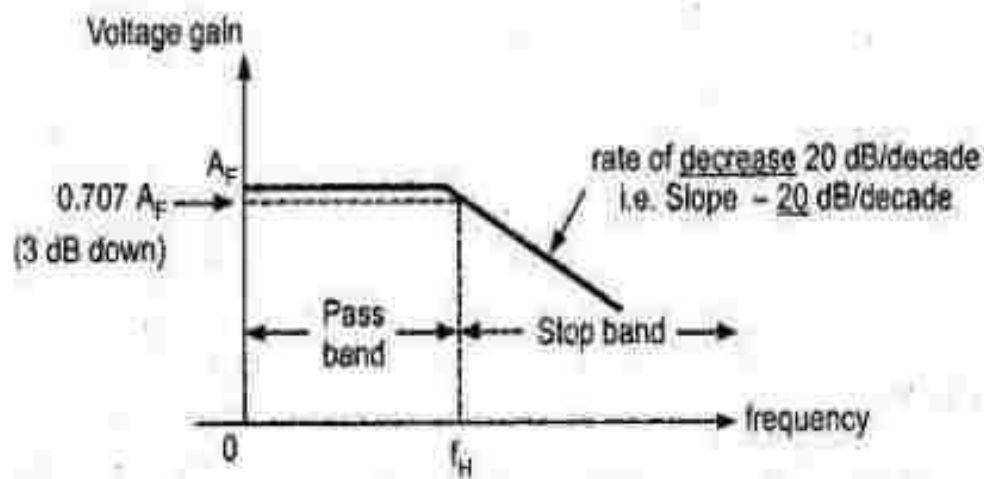


Fig. 2.75 Frequency response

Second-order Butterworth Filter

The second-order Butterworth filter consists of two reactive components. The circuit diagram of a second-order low pass Butterworth filter is as shown in the below figure.

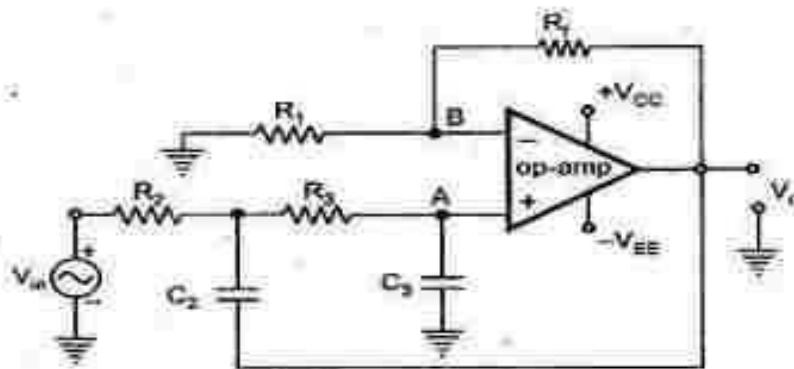


Fig. 2.76 Second order low pass butterworth filter

In this type of filter, resistor R and R_F are the negative feedback of op-amp. And the cutoff frequency of the filter decides by R_2 , R_3 , C_2 , and C_3 . The second-order lowpass Butterworth filter consists of two back-to-back connected RC networks. And R_1 is the load resistance. First-order and second-order Butterworth filters are very important. Because we can get higher-order Butterworth filter by just cascading of the first-order and second-order Butterworth filters.

Let's analyse the circuit of second-order Butterworth filter.

Apply Kirchhoff's Current Law at point V_1 .

$$I_1 = I_2 + I_3$$

$$\frac{V_{in} - V_1}{R_2} = \frac{V_1 - V_0}{\frac{1}{sC_2}} + \frac{V_1 - V_a}{R_3} \quad (1)$$

Using potential divider rule at point V_a

$$V_a = V_1 \left[\frac{\frac{1}{sC_3}}{R_3 + \frac{1}{sC_3}} \right]$$

$$V_a = V_1 \left[\frac{\frac{1}{sC_3}}{\frac{R_3 s C_3 + 1}{s C_3}} \right]$$

$$V_a = \frac{V_1}{1 + s R_3 C_3}$$

Put the value of V_1 in equation-(1) $V_1 = V_a (1 + s R_3 C_3)$

$$\begin{aligned}
\frac{V_{in} - V_a(1 + sR_3C_3)}{R_2} &= \frac{V_a(1 + sR_3C_3) - V_0}{\frac{1}{sC_2}} + \frac{V_a(1 + sR_3C_3) - V_a}{\frac{1}{R_1}} \\
\frac{V_{in}}{R_2} - \frac{V_a(1 + sR_3C_3)}{R_2} &= \frac{V_a(1 + sR_3C_3)}{\frac{1}{sC_2}} - \frac{V_0}{\frac{1}{sC_2}} + \frac{V_a(1 + sR_3C_3)}{R_3} - \frac{V_a}{R_3} \\
\frac{V_{in}}{R_2} + \frac{V_0}{\frac{1}{sC_2}} &= \frac{V_a(1 + sR_3C_3)}{\frac{1}{sC_2}} + \frac{V_a(1 + sR_3C_3)}{R_2} + \frac{V_a(1 + sR_3C_3)}{R_3} - \frac{V_a}{R_3} \\
\frac{V_{in}}{R_2} + V_0sC_2 &= V_a \left[sC_2(1 + sR_3C_3) + \frac{(1 + sR_3C_3)}{R_2} + \frac{(1 + sR_3C_3)}{R_3} - \frac{1}{R_3} \right] \\
\frac{V_{in} + V_0sC_2R_2}{R_3} &= V_a \left[\frac{R_3R_2sC_2(1 + sR_3C_3) + R_3(1 + sR_3C_3) + R_2(1 + sR_3C_3) - R_2}{R_2R_3} \right] \\
R_3(V_{in} + V_0sC_2R_2) &= V_a [R_3R_2sC_2(1 + sR_3C_3) + R_3(1 + sR_3C_3) + R_2(1 + sR_3C_3) - R_2] \\
R_3V_{in} + V_0sC_2R_2R_3 &= V_a [(1 + sR_3C_3)(R_3R_2sC_2 + R_3 + R_2) - R_2] \\
V_a &= \frac{R_3V_{in} + V_0sC_2R_2R_3}{(1 + sR_3C_3)(R_3R_2sC_2 + R_3 + R_2) - R_2}
\end{aligned}$$

Because of the non-inverting configuration of an op-amp,

$$V_0 = A_f V_a$$

Where,

$$A_f = 1 + \frac{R_f}{R_1} = \text{Gain of filter in passband}$$

$$\begin{aligned}
V_0 &= A_f \left[\frac{R_3V_{in} + V_0sC_2R_2R_3}{(1 + sR_3C_3)(R_3R_2sC_2 + R_3 + R_2) - R_2} \right] \\
V_0 \frac{A_f V_0 s C_2 R_2 R_3}{(1 + sR_3C_3)(R_3R_2sC_2 + R_3 + R_2) - R_2} &= \frac{A_f R_3 V_{in}}{(1 + sR_3C_3)(R_3R_2sC_2 + R_3 + R_2) - R_2} \\
V_0 [(1 + sR_3C_3)(R_3R_2sC_2 + R_3 + R_2) - R_2 - A_f s C_2 R_2 R_3] &= A_f R_3 V_{in} \\
\frac{V_0}{V_{in}} &= \frac{A_f R_3}{[(1 + sR_3C_3)(R_3R_2sC_2 + R_3 + R_2) - R_2 - A_f s C_2 R_2 R_3]}
\end{aligned}$$

Rearrange this equation,

$$\frac{V_0}{V_{in}} = \frac{A_f R_3}{[(1 + sR_3C_3)(R_0 + R_3 + sR_2R_3C_2) - R_2 - sA_fR_2R_3C_2]}$$

$$\frac{V_0}{V_{in}} = \frac{A_f R_3}{[(R_0 + R_3 + sR_2R_3C_2 + sR_0R_3C_3 + sR_3^2C_3 + s^2R_2R_3^2C_2C_3) - R_2 - sA_fR_2R_3C_2]}$$

$$\frac{V_0}{V_{in}} = \frac{A_f R_3}{s^2R_2R_3^2C_2C_3 + s(R_2R_3C_2 + R_0R_3C_3 + R_3^2C_3 - A_fR_2R_3C_2) + R_3}$$

$$\frac{V_0}{V_{in}} = \frac{A_f R_3}{R_2R_3^2C_2C_3 \left(s^2 + s \frac{R_2R_3C_2 + R_0R_3C_3 + R_3^2C_3 - A_fR_2R_3C_2}{R_2R_3^2C_2C_3} + \frac{R_3}{R_2R_3^2C_2C_3} \right)}$$

$$\frac{V_0}{V_{in}} = \frac{A_f}{R_2R_3C_2C_3 \left(s^2 + s \frac{R_2C_2 + R_0C_3 + R_3C_3 - A_fR_2C_2}{R_2R_3C_2C_3} + \frac{1}{R_2R_3C_2C_3} \right)}$$

$$\frac{V_0}{V_{in}} = \frac{\frac{A_f}{R_2R_3C_2C_3}}{\left(s^2 + s \frac{R_2C_2 + R_0C_3 + R_3C_3 - A_fR_2C_2}{R_2R_3C_2C_3} + \frac{1}{R_2R_3C_2C_3} \right)}$$

Compare this equation with the standard form transfer function for second-order Butterworth filter. And that is,

$$\frac{V_0}{V_{in}} = \frac{A}{s^2 + 2\zeta\omega_c s + \omega_c^2}$$

By comparing above equations, we can find the equation of cutoff frequency and overall gain for the second-order lowpass Butterworth filter.

The gain of filter is,

$$A_{max} = \frac{A_f}{R_2 R_3 C_2 C_3}$$

And the Cutoff frequency of filter is ,

$$\omega_c^2 = \frac{1}{R_2 R_3 C_2 C_3}$$

$$f_c = \frac{1}{2\pi\sqrt{R_2 R_3 C_2 C_3}}$$

Now, if we consider the value of R_2 is same as R_3 and the value of C_2 is same as C_3 .

$$R_2 = R_3 = R \quad \text{and} \quad C_2 = C_3 = C$$

$$f_c = \frac{1}{2\pi RC}$$

Now if we put above values in transfer function,

$$\frac{V_0}{V_{in}} = \frac{\frac{A_f}{R^2 C^2}}{s^2 + s \frac{RC + RC + RC - A_f RC}{R^2 C^2} + \frac{1}{R^2 C^2}}$$

$$\omega_0 = \frac{1}{RC}$$

$$\frac{V_0}{V_{in}} = \frac{A_f \omega^2}{s^2 + s(3 - A_f)\omega + \omega^2}$$

From above equation, the quality factor Q is equal to,

$$Q = \frac{1}{3 - A_f}$$

We can say that, the quality factor is only depends on the gain of filter. And the value of gain should not more than 3. If the value of gain is more than 3, the system will be unstable.

The value of quality factor is 0.707 for the Butterworth filter. And if we put this value in equation of quality factor, we can find the value of gain.

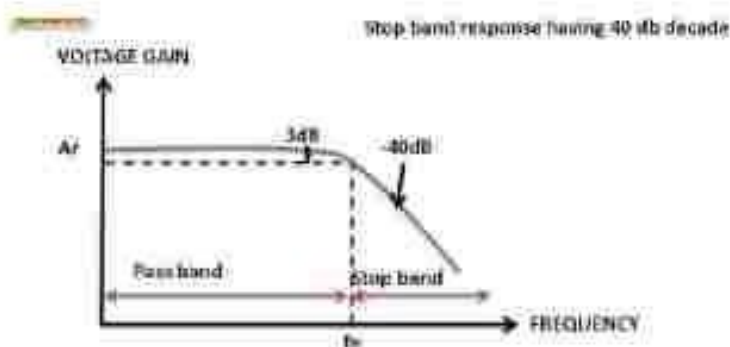
$$0.707 = \frac{1}{3 - A_f}$$

$$A_f = 1.586$$

$$1 + R_f R_1 = 1.586$$

$$R_f R_1 = 0.586$$

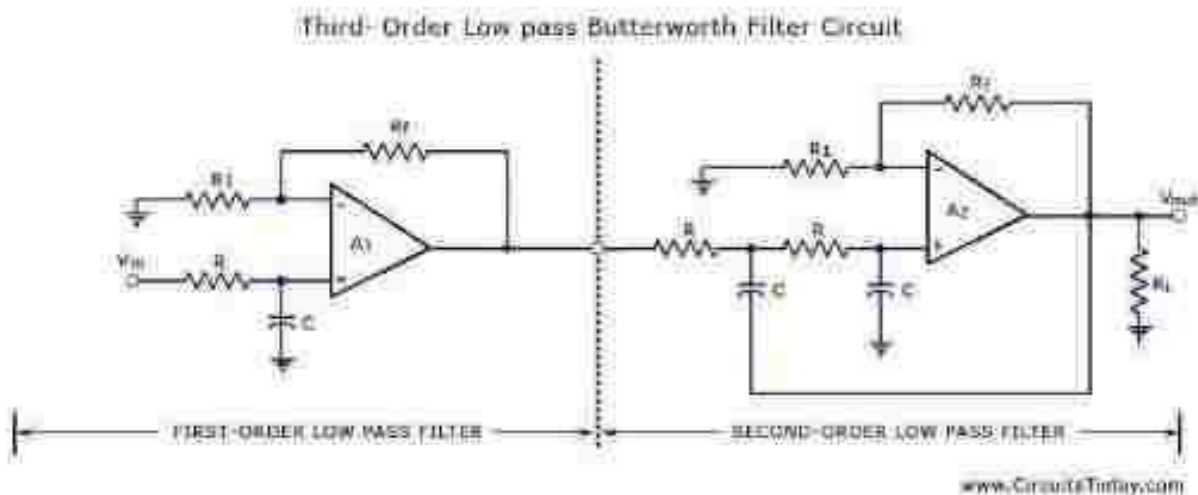
While designing the second-order Butterworth filter above relation must be satisfy. The frequency response of this filter is as shown in below figure.



Third-order Lowpass Butterworth Filter

Third-order lowpass Butterworth filter can design by cascading the first-order and second-order Butterworth filter.

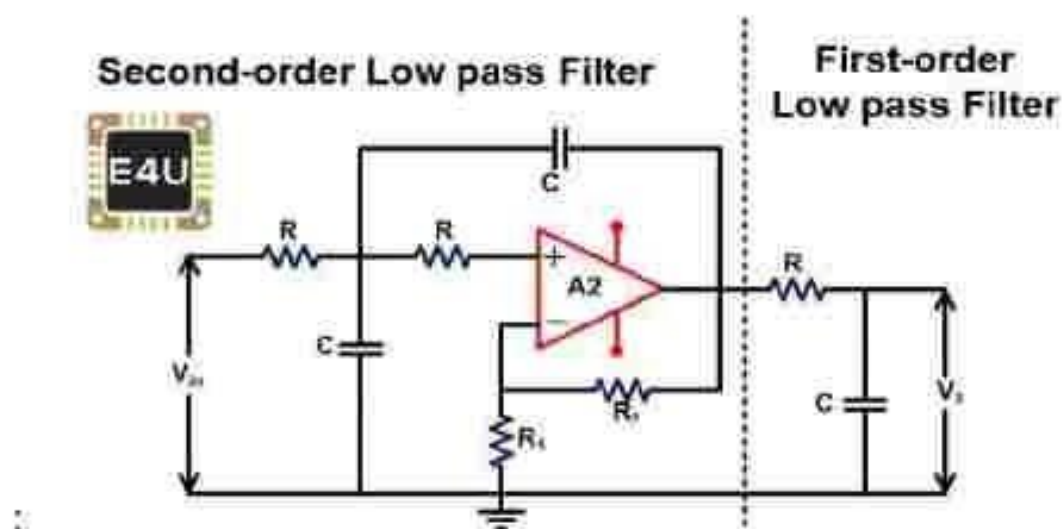
The below figure shows the circuit diagram of the third-order lowpass Butterworth filter.



Third-order Low Pass Butterworth Filter

In this figure, the first part shows the first-order lowpass Butterworth filter, and the second part shows the second-order lowpass Butterworth filter.

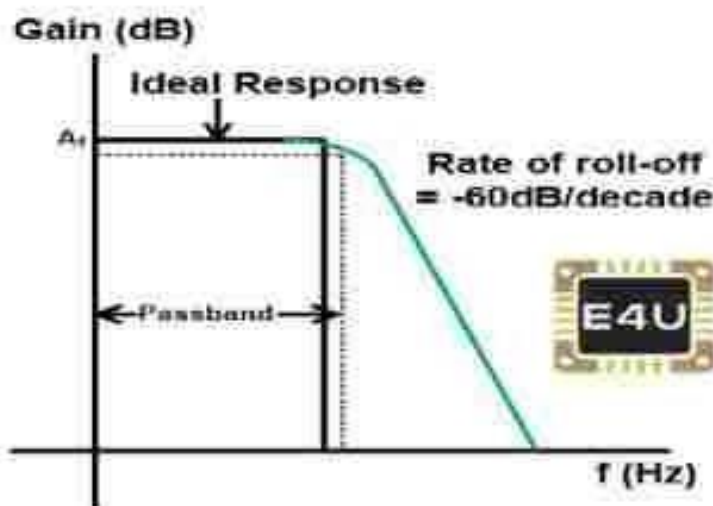
But in this condition, the voltage gain of the first part is optional and it can be set at any value. Therefore, the first op-amp is not taking part in voltage gain. Hence, the figure for the third-order low pass filter can be expressed as below figure also;



The voltage gain of a second-order filter affects the flatness of frequency response. If the gain of the second-order filter is kept at 1.586, the gain will down 3db for each part. So, the overall gain will down 6dB at the cutoff frequency.

By increasing the voltage gain of the second-order filter, we can offset the cumulative loss of voltage gain.

In the third-order Butterworth filter, the rate of a roll-off period is -60dB/decade. And the frequency response of this filter is nearer to the ideal Butterworth filter compared to the first and second-order filters. The frequ



(frequency response of this filter is as shown in the below figure.)

Fourth-order Lowpass Butterworth Filter

Fourth-order Butterworth filter is established by the cascade connection of two second-order low pass Butterworth filters. The circuit diagram of the fourth-order lowpass Butterworth filter is as shown in the below figure.